SPECTRAL MULTIPLIERS FOR LAPLACIANS WITH DRIFT ON DAMEK–RICCI SPACES

ALESSANDRO OTTAZZI AND MARIA VALLARINO

ABSTRACT. We prove a multiplier theorem for certain Laplacians with drift on Damek–Ricci spaces, which are a class of Lie groups of exponential growth. Our theorem generalizes previous results obtained by W. Hebisch, G. Mauceri and S. Meda on Lie groups of polynomial growth.

1. Introduction

W. Hebisch, G. Mauceri and S. Meda [19] studied spectral multipliers of right invariant sub-Laplacians with drift on a noncompact connected Lie group G. The operators they consider are self–adjoint with respect to a positive measure, whose density with respect to the left Haar measure is a nontrivial positive character of G. If G is amenable, they showed that every L^p spectral multiplier of such sub-Laplacians with drift extends to a bounded holomorphic function on a parabolic region in the complex plane. When G is of polynomial growth they proved that this necessary condition is nearly sufficient, by proving that bounded holomorphic functions on a suitable parabolic region which satisfy certain regularity conditions are spectral multipliers of such operators. However, if G has exponential growth, the question of finding a sufficient condition for a function to be a L^p multiplier of self-adjoint sub-Laplacians with drift remains open. In this note we contribute to this problem by considering certain Laplacians with drift on a class of Lie groups of exponential growth, namely the harmonic extensions of H-type groups.

Before going into more detail, we should mention that sub-Laplacians with drift were studied by other authors in the literature. Heat kernel estimates for sub-Laplacians with drift were studied on various Lie groups in [2, 15, 28]. N. Lohouhé and S. Mustapha [22] studied the L^p boundedness of the Riesz transforms of any order associated with sub-Laplacians with drift on every amenable Lie group. In particular, they analysed in [22, Section IV] the case of harmonic extensions of H-type groups.

Let now $(\mathcal{X},\mu)$ be a measure space and T a linear nonnegative self-adjoint operator on $L^2(\mu)$. Let $\{E(\lambda)\}$ denote the spectral resolution of the identity for which $T=\int_0^\infty \lambda \,\mathrm{d}E(\lambda)$. By the spectral theorem, if M is a bounded Borel measurable function on $[0,\infty)$, then the

---

2010 *Mathematics Subject Classification.* 47A60, 42B15, 22E30, 43A80.

Key words and phrases. Laplacian with drift, nonunimodular groups, spectral multipliers, exponential growth.

Work partially supported by Progetto GNAMPA 2012 "Analisi armonica su varietà, spazi di Wiener e gruppi di Lie" and by Progetto GNAMPA 2013 "Analisi armonica e geometrica su gruppi di Lie e spazi subRiemanniani".

operator $M(T)$ defined by

$$M(T) = \int_0^\infty M(\lambda)\,\mathrm{d}E(\lambda)$$

is bounded on $L^2(\mu)$. We call M a L^p *spectral multiplier* for T, $p \in [1,\infty) \setminus \{2\}$, if $M(T)$ extends to bounded operator on $L^p(\mu)$. The spectral multiplier problem for T consists in finding conditions, necessary or sufficient, for a function M to be a L^p multiplier for T.

We say that an operator T admits a L^p *holomorphic functional calculus* if every L^p multiplier of T extends to an holomorphic function on some neighbourhood of its L^2 spectrum. By contrast, an operator T is said to admit a L^p *differentiable functional calculus* if a function which satisfies suitable differentiable conditions on $[0,\infty)$ is a L^p multiplier for T.

The functional calculus for Laplacians on Lie groups has been intensively studied. Let G be a connected noncompact Lie group of topological dimension n. Let $X_0, ..., X_{n-1}$ be a basis of left invariant vector fields on G and let

$$L_0 = -\sum_{i=0}^{n-1} X_i^2$$

be the corresponding *Laplacian*, which is nonnegative and essentially self-adjoint on $L^2(\rho)$, where ρ is the right Haar measure of G. The functional calculus for L_0 has been the object of investigation in different classes of Lie groups. If the group G has polynomial growth, then L_0 has differentiable functional calculus [1].When G is of exponential growth the situation changes. There are classes of Lie groups of exponential growth and Laplacians L_0 which admit L^p differentiable functional calculus [10, 17, 20], and others which admit a L^p holomorphic functional calculus [6, 18, 23, 24]. We refer the reader to [10, 19] and the references therein for a more detailed discussion.

Operators of the form

$$L_X = L_0 - X\,,$$

with X a left invariant vector field, are called *Laplacians with drift* and they also have been studied by various authors. It turns out that L_X is symmetric with respect to some measure if and only if there exists a nontrivial positive character χ such that $X = \sum_{i=0}^{n-1} X_i(\chi)(e)X_i$ [19, 22], where e denotes the identity of G. In this case L_X is essentially self-adjoint on $L^2(\chi\,\mathrm{d}\rho)$ and one can study its functional calculus. If G is amenable, then the results of Hebisch, Mauceri and Meda can be reformulated in terms of left invariant vector fields and right measure and imply that L_X has L^p holomorphic functional calculus. When G has polynomial growth, they also established a sufficient condition for L^p multipliers of L_X. To the best of our knowledge, it remains unknown if a similar sufficient condition holds for exponential growth Lie groups, where the technics in [19] do not seem to apply. As we mentioned at the beginning of this section, we consider here the case of harmonic extensions of H-type groups, also called *Damek–Ricci spaces*. Such groups were introduced by E. Damek and F. Ricci [11], [12], [13], [14], and include all rank one symmetric spaces of the noncompact type. Most of them are nonsymmetric harmonic manifolds, and provide counterexamples to the Lichnerowicz conjecture. The geometry of these spaces was studied by M. Cowling, A. H. Dooley, A. Korányi and Ricci in [8], [9]. Given an H-type group N, let $S = NA$ be the one-dimensional extension of N obtained by letting $A = \mathbb{R}^+$ act on N by homogeneous dilations. The group S is solvable, hence amenable, nonunimodular and it is a Lie group of exponential growth. Let L_0 be a distinguished left-invariant Laplacian on S (see Section 3 for its precise definition). The operator L_0 has a L^p differentiable functional calculus [5, 20, 25, 27]. Our main result, Theorem 3.3, concerns spectral multipliers of L_X, where X is a drift such that L_X is symmetric. We prove that for every p in $(1,\infty)\setminus\{2\}$, a function M which is holomorphic in a parabolic region depending on the drift and on p and which satisfies suitable regularity conditions on its boundary is a L^p spectral multiplier of L_X. Our result generalizes the one proved in [19] for polynomial growth Lie groups to Damek–Ricci spaces. We prove it by splitting the kernel of the multiplier operator into a local and a global part: the analysis of the local part follows the methods in [19], while the analysis of the global part requires a different proof and is based on spherical analysis.

We conclude the introduction indicating some further questions that may be addressed. On the one hand, it would be interesting to extend the results obtained in this paper to subLaplacians with drift on Damek–Ricci spaces. On the other hand, one could also consider NA groups coming from the Iwasawa decomposition of a noncompact semisimple Lie group of arbitrary rank and study L^p multipliers of the Laplacians with drift $L_0 - X$, where L_0 is the Laplacian studied in [10] and X is a left invariant vector field such that L_X is symmetric. This will be the object of further investigation.

The paper is organized as follows: in Section 2, we recall the definition of H-type group N and its Damek–Ricci extension S; then we summarize some results of spherical analysis on S. In Section 3 we introduce the Laplacians with drift on Damek–Ricci spaces and prove our multiplier theorem.

Throughout the article the expression $A \lesssim B$ means that there exists a positive constant C such that $A \leq C\,B$.

2. Preliminaries

We recall the definition of H-type groups, describe their Damek–Ricci extensions, and collect some results on spherical analysis on these spaces that we shall use. For more details the reader is referred to [4, 5, 3, 8, 9, 11, 12, 13, 14, 16, 26].

Let $\mathfrak{n}$ be a Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$. Let $\mathfrak{v}$ and $\mathfrak{z}$ be complementary orthogonal subspaces of $\mathfrak{n}$ such that $[\mathfrak{n},\mathfrak{z}]=\{0\}$ and $[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{z}$. According to Kaplan [21], the algebra $\mathfrak{n}$ is of H-type if for every Z in $\mathfrak{z}$ of unit length the map $J_Z:\mathfrak{v}\to\mathfrak{v}$, defined by

$$\langle J_Z X, Y\rangle \,=\, \langle Z,[X,Y]\rangle \qquad \forall X,Y\in\mathfrak{v}\,,$$

is orthogonal. The connected and simply connected Lie group N associated to $\mathfrak{n}$ is called H-type group. We identify N with its Lie algebra $\mathfrak{n}$ via the exponential map

$$\begin{aligned} \mathfrak{v}\times\mathfrak{z} &\;\to\; N \\ (X,Z) &\;\mapsto\; \exp(X+Z)\,. \end{aligned}$$

Set $Q=(m_{\mathfrak{v}}+2m_{\mathfrak{z}})/2$, where $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$ are the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$, respectively.

Let S be the one-dimensional extension of N obtained as semidirect product with $A=\mathbb{R}^+$, which acts on N by homogeneous dilations. Let H denote a vector in $\mathfrak{a}$ acting on $\mathfrak{n}$ with eigenvalues $1/2$ and (possibly) 1; we extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s}=\mathfrak{n}\oplus\mathfrak{a}$, by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be a unit vector. The map

$$\begin{aligned} \mathfrak{v}\times\mathfrak{z}\times\mathbb{R}^+ &\to S \\ (X,Z,a) &\mapsto \exp(X+Z)\exp(\log a\,H) \end{aligned}$$

gives global coordinates on S. The group S is nonunimodular: the right and left Haar measures on S are given by

$$\mathrm{d}\rho(X,Z,a)=a^{-1}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a \qquad \text{and} \qquad \mathrm{d}\mu(X,Z,a)=a^{-(Q+1)}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a\,.$$

Therefore the modular function is $\delta(X,Z,a)=a^{-Q}$. We equip S with the left invariant Riemannian metric which agrees with the inner product on $\mathfrak{s}$ at the identity e. We denote by d the distance induced by this Riemannian structure. For every point $x\in S$ we write $r(x)=d(x,e)$ and we denote by $a(x)$ the component along $\mathbb{R}^+$ of x.

We may identify S with the open unit ball $\mathcal{B}$ in $\mathfrak{s}$

$$\mathcal{B}=\{(X,Z,t)\in\mathfrak{v}\times\mathfrak{z}\times\mathbb{R}:\ \|(X,Z,t)\|=|X|^2+|Z|^2+t^2<1\}\,,$$

via the bijection (see [9]) $F:S\to\mathcal{B}$ defined by

$$\begin{aligned} F(X,Z,a)=\frac{1}{\Big(1+a+\frac14|X|^2\Big)^2+|Z|^2}\bigg(\Big(1+a+\frac14|X|^2-J_Z\Big)X,2Z,\\ -1+\Big(a+\frac14|X|^2\Big)^2+|Z|^2\bigg)\,. \end{aligned}$$

The left Haar measure on S may be normalized in such a way that for all functions f in $C_c^\infty(S)$

$$\int_S f\,\mathrm{d}\mu=\int_0^\infty\int_{\partial\mathcal{B}} f\big(F^{-1}(r\omega)\big)A(r)\,\mathrm{d}r\,\mathrm{d}\sigma(\omega)\,,$$

where $\mathrm{d}\sigma$ is the surface measure on $\partial\mathcal{B}$ and

$$A(r)=2^{m_{\mathfrak{v}}+2m_{\mathfrak{z}}}\sinh^{m_{\mathfrak{v}}+m_{\mathfrak{z}}}\Big(\frac r2\Big)\cosh^{m_{\mathfrak{z}}}\Big(\frac r2\Big)\qquad\forall r\in\mathbb{R}^+\,. \tag{2.1}$$

It is easy to check that

$$A(r)\lesssim\Big(\frac{r}{1+r}\Big)^{n-1}\mathrm{e}^{Qr}\qquad\forall r\in\mathbb{R}^+\,. \tag{2.2}$$

We say that a function f on the group S is radial if it depends only on the distance from the identity, i.e., if there exists a function f_0 defined on $[0,+\infty)$ such that $f(X,Z,a)=f_0(r)$, where $r=d\big((X,Z,a),e\big)$. We abuse the notation and write $f(r)$ instead of $f_0(r)$.

Damek and Ricci [13] defined the radialisation operator

$$\mathcal{R}:\ C_c^\infty(S)\to C_c^\infty(S)$$

in the following way:

$$\mathcal{R}f(x)=\Big(\widetilde{\mathcal{R}}(f\circ F^{-1})\Big)\Big(F(x)\Big)\qquad\forall x\in S,$$

where $\widetilde{\mathcal{R}}$ is the radialisation operator on the ball $\mathcal{B}$ defined by

$$(\widetilde{\mathcal{R}}\phi)(\omega)=\frac{1}{|\partial\mathcal{B}|}\int_{\partial\mathcal{B}}\phi(\|\omega\|\omega)\,\mathrm{d}\sigma(\omega)\,.$$

A function f is radial if and only if $\mathcal{R}(f)=f$.

On the Riemannian manifold S we may consider the (positive definite) Laplace–Beltrami operator $\mathcal{L}$. A radial function ϕ on the group S is called *spherical* if it is an eigenfunction of $\mathcal{L}$ and $\phi(e) = 1$. One can prove that all spherical functions are given by $\phi_\lambda = \mathcal{R}(a(\cdot)^{-i\lambda+Q/2})$, for $\lambda \in \mathbb{C}$, and that the eigenvalue corresponding to ϕ_λ is $\lambda^2 + Q^2/4$. In [5, Lemma 1], it is shown that

$$\phi_0(r) \lesssim (1+r)\,\mathrm{e}^{-\frac{Qr}{2}} \qquad \forall r \in \mathbb{R}^+ . \tag{2.3}$$

In the following lemma we shall estimate the modulus of ϕ_λ.

Lemma 2.1. *For every $r \in \mathbb{R}^+$ and $\lambda \in \mathbb{C}$*

$$|\phi_\lambda(r)| \lesssim e^{|\mathrm{Im}\lambda| r}(1+r)e^{-\frac{Q}{2}r} .$$

Proof. For all r in $\mathbb{R}^+$

$$\begin{aligned}\phi_\lambda(r) &= \mathcal{R}\Big(a(\cdot)^{-i\lambda+\frac{Q}{2}}\Big)(x)\\ &= \frac{1}{|\partial B|}\int_{\partial B}(a\circ F^{-1})^{-i\lambda}(\|F(x)\|\omega)(a\circ F^{-1})^{\frac{Q}{2}}(\|F(x)\|\omega)d\sigma(\omega),\end{aligned} \tag{2.4}$$

where x is a point in S such that $r(x) = r$. By [3, Formula (1.20)], for every $z \in S$ we have that

$$e^{-r(z)} \le a(z) \le e^{r(z)},$$

which implies

$$a(z)^{-i\lambda} \le e^{|\mathrm{Im}\lambda| r(z)}. \tag{2.5}$$

If $\omega \in \partial B$ and $z = F^{-1}(\|F(x)\|\omega)$, then by [26, Theorem 1.1]

$$r(z) = \frac{\log(1+\|F(x)\|)}{1-\|F(x)\|} = r(x).$$

Therefore (2.5) becomes $a(z)^{-i\lambda} \le e^{|\mathrm{Im}\lambda| r(x)}$. Substituting in (2.4), we obtain

$$\begin{aligned}|\phi_\lambda(r)| &\le \frac{1}{|\partial B|}\int_{\partial B} e^{|\mathrm{Im}\lambda| r(x)}(a\circ F^{-1})^{\frac{Q}{2}}(\|F(x)\|\omega)d\sigma(\omega)\\ &= e^{|\mathrm{Im}\lambda| r(x)}\phi_0(r)\\ &\lesssim e^{|\mathrm{Im}\lambda| r}(1+r)e^{-\frac{Q}{2}r},\end{aligned}$$

where in the last inequality we used (2.3). This concludes the proof.

□

The *spherical Fourier transform* $\mathcal{H}f$ of an integrable radial function f on S is defined by the formula

$$\mathcal{H}f(\lambda) = \int_S \phi_\lambda\, f\, \mathrm{d}\mu .$$

For "nice" radial functions f on S, we have the following inversion formula

$$f(x) = c_S \int_0^\infty \mathcal{H}f(\lambda)\,\phi_\lambda(x)\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda \qquad \forall x \in S ,$$

and the Plancherel formula:

$$\int_S |f|^2 \,\mathrm{d}\mu = c_S \int_0^\infty |\mathcal{H}f(\lambda)|^2 \,|\mathbf{c}(\lambda)|^{-2} \,\mathrm{d}\lambda\,,$$

where the constant c_S depends only on $m_\mathfrak{v}$ and $m_\mathfrak{z}$, and $\mathbf{c}$ denotes the Harish-Chandra function (see [3, Section 2]). Let $\mathcal{A}$ denote the Abel transform defined for any radial function f on S by

$$\mathcal{A}f(t) = \int_N f(X, Z, \mathrm{e}^t)\,\mathrm{e}^{-Qt/2}\,\mathrm{d}X\,\mathrm{d}Z \qquad \forall t \in \mathbb{R}\,,$$

and let $\mathcal{F}$ denote the Fourier transform on the real line, defined by

$$\mathcal{F}g(s) = \int_{-\infty}^{+\infty} g(r)\,\mathrm{e}^{-isr}\,\mathrm{d}r\,,$$

for each integrable function g on $\mathbb{R}$. It is well known that $\mathcal{H} = \mathcal{F}\circ\mathcal{A}$, hence $\mathcal{H}^{-1} = \mathcal{A}^{-1}\circ\mathcal{F}^{-1}$. For later use, we recall the inversion formula for the Abel transform [3, Formula (2.24)]. If $m_\mathfrak{z}$ is even, then

$$\mathcal{A}^{-1}f(r) = a_S^e \Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^{m_\mathfrak{z}/2}\Big(-\frac{1}{\sinh(r/2)}\frac{\partial}{\partial r}\Big)^{m_\mathfrak{v}/2} f(r)\,, \tag{2.6}$$

where $a_S^e = 2^{-(3m_\mathfrak{v}+m_\mathfrak{z})/2}\pi^{-(m_\mathfrak{v}+m_\mathfrak{z})/2}$. On the other hand, if $m_\mathfrak{z}$ is odd, then

$$\mathcal{A}^{-1}f(r) = a_S^o \int_r^\infty \Big(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\Big)^{(m_\mathfrak{z}+1)/2}\Big(-\frac{1}{\sinh(s/2)}\frac{\partial}{\partial s}\Big)^{m_\mathfrak{v}/2} f(s)\,\mathrm{d}\nu(s)\,, \tag{2.7}$$

where $a_S^o = 2^{-(3m_\mathfrak{v}+m_\mathfrak{z})/2}\pi^{-n/2}$ and $\mathrm{d}\nu(s) = (\cosh s - \cosh r)^{-1/2}\sinh s\,\mathrm{d}s$.

In the sequel for every p in $(1,\infty)$ we shall denote by $Cv_p(\rho)$ the space of right $L^p(\rho)$ convolutors on S, i.e. the space of distributions k on S such that the operator $f \mapsto f * k$ is bounded on $L^p(\rho)$.

3. Laplacians with drift on Damek–Ricci spaces

Let $X_0, X_1, \ldots, X_{n-1}$ be a frame of orthonormal left invariant vector fields on S such that $X_0(e) = H$, $\{X_1(e), \ldots, X_{m_\mathfrak{v}}(e)\}$ is an orthonormal basis of $\mathfrak{v}$ and $\{X_{m_\mathfrak{v}+1}(e), \ldots, X_{n-1}(e)\}$ is an orthonormal basis of $\mathfrak{z}$. Let L_0 be the corresponding Laplacian $L_0 = -\sum_{i=0}^{n-1} X_i^2$.

The operator L_0 has the following relationship with the Laplace–Beltrami operator $\mathcal{L}$ (see [4, Proposition 2])

$$\delta^{-1/2} L_0\,\delta^{1/2} f = \Big(\mathcal{L} - \frac{Q^2}{4}\Big) f, \tag{3.1}$$

for all smooth compactly supported radial functions f on S. The spectra of $\mathcal{L} - Q^2/4$ on $L^2(\mu)$ and L_0 on $L^2(\rho)$ are both $[0,+\infty)$. For each bounded measurable function m on $\mathbb{R}^+$ the operators $m(\mathcal{L} - Q^2/4)$ and $m(L_0)$ are spectrally defined and related by

$$\delta^{-1/2} m(L_0)\,\delta^{1/2} f = m(\mathcal{L} - Q^2/4) f\,,$$

for smooth compactly supported radial functions f on S. If $k_{m(L_0)}$ and $k_{m(\mathcal{L}-Q^2/4)}$ denote the convolution kernels of $m(L_0)$ and $m(\mathcal{L} - Q^2/4)$, then

$$m(\mathcal{L} - Q^2/4) f = f * k_{m(\mathcal{L}-Q^2/4)} \qquad \text{and} \qquad m(L_0) f = f * k_{m(L_0)} \qquad \forall f \in C_c^\infty(S)\,,$$

where $*$ denotes the convolution on S. We shall also write $m(L_0)\delta_e$ to denote $k_{m(L_0)}$. The following proposition is proved in [4, 3].

Proposition 3.1. *Let m be a bounded measurable function on $\mathbb{R}^+$. Then $k_{m(\mathcal{L}-Q^2/4)}$ is radial and $k_{m(L_0)} = \delta^{1/2}\, k_{m(\mathcal{L}-Q^2/4)}$. The spherical transform $\mathcal{H}k_{m(\mathcal{L}-Q^2/4)}$ of $k_{m(\mathcal{L}-Q^2/4)}$ is given by*

$$\mathcal{H}k_{m(\mathcal{L}-Q^2/4)}(\lambda) = m(\lambda^2),$$

for every positive real number λ.

Nontrivial positive characters of S are given by $\chi_\alpha(X, Z, a) = a^\alpha$, $\alpha \in \mathbb{R} \setminus \{0\}$. Let us consider the left invariant vector field $X_\alpha = \alpha\, X_0$ and suppose that $\alpha \neq 0$. Reformulating the result by Hebisch, Mauceri and Meda [19, Proposition 3.1] in terms of left invariant vector fields and right measure, we get that the Laplacian with drift $L_X = L_0 - X$ is symmetric with respect to some measure if and only if $X = \alpha X_0$. We shall denote by L_α the Laplacian with drift

$$L_\alpha = L_0 - \alpha X_0\,.$$

It is essentially self-adjoint on $L^2(\rho_\alpha)$, where $\mathrm{d}\rho_\alpha = \chi_\alpha\ \mathrm{d}\rho$ and its spectrum is contained in the interval $[\alpha^2/4, \infty)$.

Remark 3.2. A direct computation of the Laplace–Beltrami operator of Damek–Ricci spaces gives

$$\mathcal{L} = L_0 + QX_0 = L_{-Q}\,.$$

Thus the functional calculus for L_{-Q} is equivalent to the functional calculus for the Laplace–Beltrami operator on S, which was studied in [5, 7].

Let M be a bounded measurable function on $[\alpha^2/4, \infty)$. We shall give a sufficient condition on a function M to be a L^p spectral multiplier of L_α. To do so, let us introduce some notation. For every $p \in (1, \infty) \setminus \{2\}$ let $P_{\alpha,p}$ be the parabolic region

$$P_{\alpha,p} = \left\{ x + iy \in \mathbb{C} :\ x > \frac{y^2}{\alpha^2\,\sin^2\phi_p^*} + \alpha^2/4\,\cos^2\phi_p^* \right\}\,,$$

where $\phi_p^* = \arcsin|2/p - 1|$. For every positive numbers W and β we denote by Σ_W the complex strip defined by $\Sigma_W = \{x + iy \in \mathbb{C} : |y| < W\}$ and by $H^\infty(\Sigma_W; \beta)$ the space of bounded holomorphic functions f on the strip Σ_W such that

$$|D^j f(s \pm iW)| \leq C\,(1 + s^2)^{-j/2} \qquad \forall j = 0, ..., \beta, \forall s \in \mathbb{R}\,.$$

We shall denote by $W_{\alpha,p}$ the number $|\alpha|\,|1/p - 1/2|$. Our main result is the following.

Theorem 3.3. *Let $p \in (1, \infty) \setminus \{2\}$. Suppose that $M \in H^\infty(P_{\alpha,p})$ and that the function M_α defined by $M_\alpha(z) = M(z^2 + \alpha^2/4)$ lies in $H^\infty(\Sigma_{W_{\alpha,p}}; \beta)$, with $\beta > \max(2, n/2)$. Then $M(L_\alpha)$ extends to a bounded operator on $L^p(\rho_\alpha)$.*

Proof. By [19, Proposition 4.1] for each p in $(1, \infty)$ the operator $M(L_\alpha)$ is bounded on $L^p(\rho_\alpha)$ if and only if $\chi_\alpha^{1/p-1/2}\, M(L_0 + \alpha^2/4)\delta_e$ is in $Cv_p(\rho)$.

Let ω be a smooth cutoff function supported in $[-1, 1]$, equals 1 in $[-1/4, 1/4]$ such that

$$\sum_{h\in\mathbb{Z}} \omega(t - h) = 1 \qquad \forall t \in \mathbb{R}\,.$$

For all $h \geq 2$, let

$$\omega_h(t) = \omega(t-h+1) + \omega(t+h-1).$$

Note that $\operatorname{supp} \omega_h \subset [h-2,h] \cup [-h,-h+2]$. We split the kernel of $M(L_0+\alpha^2/4)$ into a local and global part as follows:

$$M(L_0+\alpha^2/4)\delta_e = \hat{\eta} *_{\mathbb{R}} M_\alpha(\sqrt{L_0})\delta_e + \widehat{(1-\eta)} *_{\mathbb{R}} M_\alpha(\sqrt{L_0})\delta_e = \ell + g\,,$$

where $\eta = \omega + \omega_2$. Define

$$\ell_p = \chi_\alpha^{1/p-1/2}\ell \qquad \text{and} \qquad g_p = \chi_\alpha^{1/p-1/2} g\,. \tag{3.2}$$

In Proposition 3.5 and Proposition 3.8, we shall prove that $\ell_p \in Cv_p(\rho)$ and $g_p \in Cv_p(\rho)$, respectively. This concludes the proof. □

As a direct consequence of the theorem above we compute the spectrum of L_α on $L^p(\rho_\alpha)$, which we denote by $\sigma_p(L_\alpha)$.

Corollary 3.4. *Let $p \in (1,\infty) \setminus \{2\}$. The spectrum $\sigma_p(L_\alpha)$ is the parabolic region $P_{\alpha,p}$.*

Proof. If $w \notin P_{\alpha,p}$, then the function $M(z) = (w-z)^{-1}$ satisfies the hypothesis of Theorem 3.3. Then $(w-L_\alpha)^{-1}$ is bounded on $L^p(\rho_\alpha)$ and w belongs to the resolvent of L_α on $L^p(\rho_\alpha)$. This proves that $\sigma_p(L_\alpha) \subseteq P_{\alpha,p}$.

On the other hand, let $w \in P_{\alpha,p}$. Then the function $M(z) = (w-z)^{-1}$ is not holomorphic in $P_{\alpha,p}$. By [19, Theorem 4.2] the operator $M(L_\alpha) = (w-L_\alpha)^{-1}$ is not bounded on $L^p(\rho_\alpha)$. Then $w \in \sigma_p(L_\alpha)$. □

We proceed next with the analysis of the local and global part of the kernel of $M(L_0 + \alpha^2/4)$ constructed above separately. In particular, we note that on Lie groups of polynomial growth [19] the analysis of the global part of the kernel was based on the ultracontractivity of the heat semigroup generated by L_0 and a consequence of the Dunford–Pettis Theorem. However, since Damek–Ricci spaces are nonunimodular, the heat semigroup associated with L_0 is not ultracontractive, i.e., e^{-tL_0} is not bounded from $L^1(\rho)$ to $L^\infty(\rho)$. Moreover, Dunford–Pettis Theorem cannot be applied as in [19]. To study the behavior of the global part of the kernel we shall instead use tools from spherical analysis.

3.1. **Analysis of the local part.** Note that by the Fourier inversion formula

$$\ell = \hat{\eta} *_{\mathbb{R}} M_\alpha(\sqrt{L_0})\delta_e = \Big(\frac{1}{2\pi}\int_{\mathbb{R}} \eta(t)\, \hat{M}_\alpha(t)\, \cos(t\sqrt{L_0})\, \mathrm{d}t\Big)\delta_e\,.$$

Since η is supported in $[-2,2]$, by finite propagation speed, ℓ is supported in $B(e,2)$.

We recall that a function $M : \mathbb{R}^+ \to \mathbb{C}$ satisfies a mixed Mihlin–Hörmander condition of order (s_0, s_∞) if

$$\max_{j=0,\dots,s_0} \sup_{0<v<1} |v^j\, D^j M(v)| < \infty\,,$$

and

$$\max_{j=0,\dots,s_\infty} \sup_{v\geq 1} |v^j\, D^j M(v)| < \infty\,.$$

In this case we say that $M \in \operatorname{Horm}(s_0, s_\infty)$.

Proposition 3.5. *The following hold:*

(i) *Let* $s_0 > 3/2$ *and* $s_\infty > \max(3/2, n/2)$*. If* $M_\alpha \in \mathrm{Horm}(s_0, s_\infty)$*, then* $\hat\eta *_{\mathbb{R}} M_\alpha(\sqrt{L_0})$ *is of weak type* 1 *and bounded on* $L^r(\rho)$ *for all* $r \in (1,\infty)$*;*

(ii) *given* $p \in (1,\infty) \setminus 2$*, the function* $\chi_\alpha^{(1/p-1/2)}\, \hat\eta *_{\mathbb{R}} M_\alpha(\sqrt{L_0})\delta_e$ *is in* $Cv_p(\rho)$*.*

Proof. As in [19, Proposition 5.3] we prove that if $M_\alpha \in \mathrm{Horm}(s_0, s_\infty)$, then $v \mapsto \hat\eta *_{\mathbb{R}} M_\alpha(\sqrt{v})$ lies in $\mathrm{Horm}(s_0, s_\infty)$. By [27] this implies that $\hat\eta *_{\mathbb{R}} M_\alpha(\sqrt{L_0})$ is of weak type 1 and bounded on $L^r(\rho)$ for all $r \in (1,\infty)$. This implies tht $\ell = \hat\eta *_{\mathbb{R}} M_\alpha(\sqrt{L_0})\delta_e$ is in $Cv_r(\rho)$ for all $r \in (1,\infty)$.

Since the function ℓ is supported in $B(e,2)$ and $Cv_p(\rho)$ is a $C_c^\infty(S)$ module, we deduce that the function $\ell_p = \chi_\alpha^{(1/p-1/2)}\, \ell$ is also in $Cv_p(\rho)$.

□

Remark 3.6. Notice that the condition we require on M_α in Proposition 3.5 is weaker than the condition required in the statement of Theorem 3.3. The holomorphy of the multiplier is only used in the study of the global part of the kernel.

3.2. **Analysis of the global part.** We decompose the global part of the kernel g_p as

$$g_p = \chi_\alpha^{(1/p-1/2)} \sum_{h=3}^{\infty} P_h(\sqrt{L_0})\delta_e = \sum_{h=3}^{\infty} g_{p,h}\,, \tag{3.3}$$

where $\hat P_h = \omega_h\, \hat M_\alpha$. By Proposition 3.1 $P_h(\sqrt{L_0})\delta_e = \delta^{1/2}\, k_h$, where k_h is the radial function whose spherical transform is $\mathcal{H}k_h = P_h$. We need the following estimates of the derivatives of the functions $\hat P_h$.

Lemma 3.7. *Let* M_α *be in* $H^\infty(\Sigma_{W_{\alpha,p}};\beta)$ *and* P_h *be defined as above. For every* $p, q \in \mathbb{N}$ *such that* $1 \le p+q \le \beta$ *we have*

$$\Big|\Big(-\frac{1}{\sinh s}\partial_s\Big)^p\Big(-\frac{1}{\sinh s}\partial_s\Big)^q \hat P_h(s)\Big| \lesssim e^{-(p+\frac{q}{2})\,s}\, s^{-\beta}\, e^{-W_{\alpha,p}\,s} \qquad \forall s > 1\,.$$

Proof. By an induction argument, we may prove that there exist smooth functions $\psi_1, \ldots, \psi_{p+q}$ bounded and with bounded derivatives of any order in $(1,+\infty)$, such that

$$\Big(-\frac{1}{\sinh s}\partial_s\Big)^p\Big(-\frac{1}{\sinh s}\partial_s\Big)^q \hat P_h(s) = e^{-(p+\frac{q}{2})\,s} \sum_{i=1}^{p+q} \psi_i(s) D^i \hat P_h(s) \qquad \forall s > 1\,. \tag{3.4}$$

Following the argument of [19, Proposition 5.8] we see that since M_α is in $H^\infty(\Sigma_{W_{\alpha,p}};\beta)$, then

$$|D^k \hat P_h(t)| \le C\, \|M_\alpha\|_{W_{\alpha,p};\beta}\, (1+|t|)^{-\beta}\, \mathrm{e}^{-W_{\alpha,p}|t|} \qquad \forall t \in (h-2,h),\ \forall k \le \beta\,, \tag{3.5}$$

and $D^k \hat P_h(t) = 0$ if $t \notin (h-2,h)$.

We then apply estimate (3.5) in formula (3.4) to complete the proof. □

Proposition 3.8. *If* M_α *is in* $H^\infty(\Sigma_{W_{\alpha,p}};\beta)$ *for* $\beta > 2$*, then the function* g_p *defined in* (3.2) *is in* $L^1(\rho)$*. In particular,* g_p *lies in* $Cv_p(\rho)$ *for all* $p \in (1,\infty)$*.*

Proof. We compute the L^1 norm of each function $g_{p,h}$:

$$
\begin{aligned}
\|g_{p,h}\|_{L^1(\rho)} &= \int \chi_\alpha^{(1/p-1/2)}\ |P_h(\sqrt{L_0})\delta_e|\,\mathrm{d}\rho \\
&= \int a^{\alpha(1/p-1/2)}\,\delta^{1/2}\,|k_h|\,\mathrm{d}\rho \\
&= \int a^{\alpha(1/p-1/2)}\,\delta^{-1/2}\,|k_h|\,\mathrm{d}\mu \\
&= \int \delta^{-\frac{\alpha}{Q}(1/p-1/2)-1/2}\,|k_h|\,\mathrm{d}\mu \\
&= \int \phi_{i\left(\alpha(1/p-1/2)\right)}\,|k_h|\ \mathrm{d}\mu \\
&= \int_0^\infty \phi_{i\left(\alpha(1/p-1/2)\right)}(r)\,|k_h(r)|\,A(r)\,\mathrm{d}r\,,
\end{aligned}
\tag{3.6}
$$

where $A(r)$ is defined in formula (2.1).

The kernel k_h can be computed by using the inverse Abel transform. We study the cases when $m_{\mathfrak{z}}$ is either even or odd.

Case $m_{\mathfrak{z}}$ even. In this case by formula (2.6)

$$k_h(r) = \mathcal{A}^{-1}\mathcal{F}^{-1}P_h(r) = a_S^e\Big(-\frac{1}{\sinh r}\partial_r\Big)^{m_{\mathfrak{z}}/2}\Big(-\frac{1}{\sinh r/2}\partial_r\Big)^{m_{\mathfrak{v}}/2}\hat{P}_h(r)\,.$$

Clearly $k_h(r)=0$ if $r\notin(h-2,h)$. By Lemma 3.7 if $r\in(h-2,h)$, then

$$|k_h(r)| \lesssim e^{-\frac{Q}{2}r}\,r^{-\beta}\,e^{-W_{\alpha,p}\,r}\,.$$

Then by (3.6) we have

$$
\begin{aligned}
\|g_{p,h}\|_{L^1(\rho)} &\lesssim \int_{h-2}^h e^{W_{\alpha,p}\,r}\,e^{-\frac{Q}{2}r}\,(1+r)\,e^{-\frac{Q}{2}r}\,r^{-\beta}\,e^{-W_{\alpha,p}\,r}\,e^{Q\,r}\ \mathrm{d}r \\
&\lesssim h^{1-\beta}\,,
\end{aligned}
$$

where we applied Lemma 2.1 with $\lambda = i\alpha(1/p-1/2)$ and (2.2). By summing over $h\geq 3$ we obtain that $g_p\in L^1(\rho)$ if $\beta>2$.

Case $m_{\mathfrak{z}}$ odd. In this case by formula (2.7)

$$k_h(r) = \mathcal{A}^{-1}\mathcal{F}^{-1}P_h(r) = a_S^o\int_r^\infty\Big(-\frac{1}{\sinh s}\partial_s\Big)^{\frac{m_{\mathfrak{z}}+1}{2}}\Big(-\frac{1}{\sinh s/2}\partial_s\Big)^{\frac{m_{\mathfrak{v}}}{2}}\hat{P}_h(s)\,\frac{\sinh s}{\sqrt{\cosh s-\cosh r}}\ \mathrm{d}s\,.$$

If $r \geq h$, then $k_h(r) = 0$. If $r \in (h-2, h)$, then by Lemma 3.7

$$
\begin{aligned}
|k_h(r)| &\lesssim \int_r^h e^{-\frac{Q+1}{2}\,s}\, s^{-\beta}\, e^{-W_{\alpha,p}\,s}\, \frac{e^s}{\sqrt{\cosh s - \cosh r}}\,\mathrm{d}s \\
&\lesssim \int_r^h e^{-\frac{Q+1}{2}\,s}\, s^{-\beta}\, e^{-W_{\alpha,p}\,s}\, \frac{e^s}{e^{r/2}\sqrt{s-r}}\,\mathrm{d}s \\
&\lesssim e^{-\frac{Q+1}{2} r}\, r^{-\beta}\, e^{-W_{\alpha,p}\,r}\, e^{r/2} \int_0^2 \frac{\mathrm{d}v}{\sqrt{v}} \\
&\lesssim e^{-\frac{Q}{2}\,r}\, r^{-\beta}\, e^{-W_{\alpha,p}\,r}\,.
\end{aligned}
$$

If $r < h-2$, then by Lemma 3.7

$$
\begin{aligned}
|k_h(r)| &\lesssim \int_{h-2}^h e^{-\frac{Q+1}{2}\,s}\, s^{-\beta}\, e^{-W_{\alpha,p}\,s}\, \frac{e^s}{e^{r/2}\sqrt{s-r}}\,\mathrm{d}s \\
&\lesssim e^{-\frac{Q+1}{2}\,h}\, h^{-\beta}\, e^{-W_{\alpha,p}\,h}\, e^h e^{-r/2} \int_{h-2}^h \frac{1}{\sqrt{s-r}}\,\mathrm{d}s \\
&\lesssim e^{-\frac{Q+1}{2}\,h}\, h^{-\beta}\, e^{-W_{\alpha,p}\,h}\, e^h e^{-r/2} \frac{1}{\sqrt{h-2-r}}.
\end{aligned}
$$

By (3.6), Lemma 2.1, (2.2) and the above estimates, we have

$$
\begin{aligned}
\|g_{p,h}\|_{L^1(\rho)} \lesssim& \int_0^1 e^{-\frac{Q-1}{2}\,h}\, h^{-\beta-\frac{1}{2}}\, e^{-W_{\alpha,p}\,h}\, r^{n-1}\,\mathrm{d}r \\
&+ \int_1^{h-2} e^{W_{\alpha,p}\,r}\, e^{-\frac{Q}{2}\,r}\,(1+r)\, e^{-\frac{Q+1}{2}\,h}\, h^{-\beta}\, e^{-W_{\alpha,p}\,h}\, e^h e^{-r/2} \frac{1}{\sqrt{h-2-r}}\, e^{Q\,r}\,\mathrm{d}r \\
&+ \int_{h-2}^h e^{W_{\alpha,p}\,r}\, e^{-\frac{Q}{2}\,r}\,(1+r)\, e^{-\frac{Q}{2}\,r}\, r^{-\beta}\, e^{-W_{\alpha,p}\,r}\, e^{Q\,r}\,\mathrm{d}r \\
\lesssim& \, e^{-\frac{Q-1}{2}\,h}\, h^{-\beta-\frac{1}{2}}\, e^{-W_{\alpha,p}\,h} + \mathrm{I} + h^{1-\beta}.
\end{aligned}
$$

We are left with the estimate of the summand

$$
\mathrm{I} = \int_1^{h-2} e^{W_{\alpha,p}\,r}\, e^{-\frac{Q}{2}\,r}\,(1+r)\, e^{-\frac{Q+1}{2}\,h}\, h^{-\beta}\, e^{-W_{\alpha,p}\,h}\, e^h e^{-r/2} \frac{1}{\sqrt{h-2-r}}\, e^{Q\,r}\,\mathrm{d}r.
$$

By the change of variables $v = h-2-r$, we get

$$
\begin{aligned}
I &\lesssim h^{1-\beta} e^{-\frac{Q-1}{2}h} e^{-W_{\alpha,p}h} \int_0^{h-3} \frac{e^{(\frac{Q-1}{2}+W_{\alpha,p})(h-2-v)}}{\sqrt{v}}\,\mathrm{d}v \\
&\lesssim h^{1-\beta}.
\end{aligned}
$$

Therefore $\|g_{p,h}\|_{L^1(\rho)} \lesssim h^{1-\beta}$. By summing over $h \geq 3$ we obtain that $g_p \in L^1(\rho)$ if $\beta > 2$. □

References

[1] G. Alexopoulos, *Spectral multipliers on Lie groups of polynomial growth*. Proc. Amer. Math. Soc. 120 (1994), 973–979

[2] G. Alexopoulos, *Sub-Laplacians with drift on Lie groups of polynomial volume growth*. Mem. Amer. Math. Soc. 155 (2002), no. 739, x+101 pp.

[3] J.-Ph. Anker, E. Damek, C. Yacoub, *Spherical analysis on harmonic AN groups*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1996), 643–679
[4] F. Astengo, *The maximal ideal space of a heat algebra on solvable extensions of H-type groups*. Boll. Un. Mat. Ital. A(7) **9** (1995), 157–165
[5] F. Astengo, *Multipliers for a distinguished Laplacean on solvable extensions of H-type groups*. Monatsh. Math. **120** (1995), 179–188
[6] M. Christ, D. Müller, *On Lp spectral multipliers for a solvable Lie group*. Geom. Funct. Anal. 6 (1996), 860–876
[7] J.L. Clerc, E.M. Stein, *Lp-multipliers for noncompact symmetric spaces*. Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 3911–3912.
[8] M. Cowling, A. Dooley, A. Korányi, F. Ricci, *H-type groups and Iwasawa decompositions*. Adv. Math. **87** (1991), 1–41
[9] M. Cowling, A. Dooley, A. Korányi, F. Ricci, *An approach to symmetric spaces of rank one via groups of Heisenberg type.* J. Geom. Anal. **8** (1998), 199–237
[10] M. Cowling, S. Giulini, A. Hulanicki, G. Mauceri, *Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth*. Studia Math. **111** (1994), 103–121
[11] E. Damek, *Curvature of a semidirect extension of a Heisenberg type nilpotent group*. Colloq. Math. **53** (1987), 249–253
[12] E. Damek, *Geometry of a semidirect extension of a Heisenberg type nilpotent group.* Colloq. Math. **53** (1987), 255–268
[13] E. Damek, F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*. Bull. Amer. Math. Soc. **27** (1992), 139–142
[14] E. Damek, F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*. J. Geom. Anal. **2** (1992), 213–248
[15] N. Dungey, *Heat kernel and semigroup estimates for sublaplacians with drift on Lie groups*. Publ. Mat. **49** (2005), 375–391
[16] J. Faraut, *Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques*. Les Cours du C.I.M.P.A. (1983)
[17] W. Hebisch, *Spectral multipliers on exponential growth solvable Lie groups*. Math. Z. 229 (1998), 435–441
[18] W. Hebisch, J. Ludwig, D. Müller, *Sub-Laplacians of holomorphic Lp-type on exponential solvable groups*. J. London Math. Soc. (2) 72 (2005), 364–390
[19] W. Hebisch, G. Mauceri, S. Meda, *Spectral multipliers for sub-Laplacians with drift on Lie groups*. Math. Z. **251** (2005), 899–927
[20] W. Hebisch, T. Steger, *Multipliers and singular integrals on exponential growth groups*. Math. Z. **245** (2003), 37–61
[21] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*. Trans. Amer. Math. Soc. **258** (1975), 145–159
[22] N. Lohoué, S. Mustapha, *Sur les transformées de Riesz dans le cas du Laplacien avec drift*. Trans. Amer. Math. Soc. **356** (2004), 2139–2147
[23] J. Ludwig, D. Müller, *Sub-Laplacians of holomorphic Lp-type on rank one AN-groups and related solvable groups*. J. Funct. Anal. 170 (2000), 366–427
[24] J. Ludwig, D. Müller, S. Souaifi, *Holomorphic Lp-type for sub-Laplacians on connected Lie groups*. J. Funct. Anal. 255 (2008), 1297–1338
[25] S. Mustapha, *Multiplicateurs spectraux sur certains groupes non-unimodulaires*. Harmonic analysis and number theory (Montreal, PQ, 1996), 11–30, CMS Conf. Proc., 21, Amer. Math. Soc., Providence, RI, 1997
[26] F. Ricci, *The spherical transform on harmonic extensions of H-type groups*. Differential geometry (Turin, 1992). Rend. Sem. Mat. Univ. Politec. Torino **50** (1992), 381–392
[27] M. Vallarino, *Spectral multipliers on Damek-Ricci spaces*. J. Lie Theory **17** (2007), 163–189
[28] N. Th. Varopoulos, L. Saloff-Coste, T. Coulhon, *Analysis and geometry on groups*. Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992

CIRM Fondazione Bruno Kessler, Via Sommarive 14, 38123 Trento, Italy
E-mail address: ottazzi@fbk.eu

Dipartimento di Scienze Matematiche Giuseppe Luigi Lagrange, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail address: maria.vallarino@polito.it