L_p-FUNCTIONAL AND J-HOLOMORPHIC CURVES IN KÄHLER MANIFOLDS

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Abstract. In this note, we give another variational characterization of J_M-holomorphic curves in Kähler manifolds, generalizing our results in [1]. We prove that if the functional L_p has a critical point or a stable point in the fixed Kähler class, then the immersion is J_M-holomorphic or Lagrangian.

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1. Introduction

Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact Kähler manifold of complex dimension n with Kähler form $\bar{\omega}$, compatible complex structure J_M, and associated Riemannian metric $\bar{g}$. Namely, for $U, V \in TM$,

$$\bar{g}(U, V) = \bar{\omega}(U, J_M V). \tag{1.1}$$

Let Σ be a closed connected real surface and $F : \Sigma \to M$ be an immersion. When an immersion is J_M-holomorphic is an interesting question in differential geometry. Recently, we ([1]) gave a variational characterization of J_M-holomorphic curves in symplectic manifolds. More precisely, we consider the change of the area functional according to the change of the symplectic form on M in the fixed cohomology class (with fixed immersion F and fixed almost complex structure J_M on M). Our first result (Theorem 2.4 of [1]) says that if the area functional has a critical point, then the immersion is J_M-holomorphic. For the stable case, our second theorem (Theorem 3.2 of [1]) says that, if the area functional has a compatible stable point, then the immersion is also J_M-holomorphic. The area functional is a natural candidate to be considered because for a J_M-holomorphic immersion, the area functional is constant in the same cohomology class (Proposition 2.2 of [1]). Thus every point is both a critical point and stable point for a J_M-holomorphic curve. Recently, we generalized the results in [1] to arbitrary dimension and codimension as well as current case ([2]).

In this note, we consider a family of more general functionals defined in terms of the Kähler angle. Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact Kähler manifold as above. Recall that the Kähler angle α of a surface Σ^2 in M is defined by ([3])

$$\bar{\omega}|_\Sigma = \cos\alpha d\mu_\Sigma, \tag{1.2}$$

where $d\mu_\Sigma$ is the induced volume form on Σ. We call an immersion $F : \Sigma \to M$ *symplectic* if $\cos\alpha > 0$ and *Lagrangian* if $\cos\alpha \equiv 0$. For a given immersion $F : \Sigma \to (M^{2n}, \bar{\omega}, J_M, \bar{g})$,

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we can define the following functional

$$L_p = \int_\Sigma \cos^p \alpha d\mu_g, \quad p \in \mathbb{Z}. \tag{1.3}$$

When $p < 0$, we assume the immersion to be symplectic in order to guarantee that the integral makes sense. When $p = 0$, L_0 is just the area functional.

When $p = -1$, by fixing the Kähler metric on M and moving the surface, Han-Li ([4]) computed the Euler-Lagrangian equation for the critical point of the functional L_{-1}, which they called *symplectic critical surface*. They also studied some properties of symplectic critical surface. For example, similar to symplectic minimal surfaces, they proved that if M is a Kähler-Einstein surface with nonnegative scalar curvature, then every symplectic critical surface is holomorphic. Later on, Han-Li ([5]) computed the second variation formula for the functional L_{-1}, and proved some rigidity results on stable symplectic critical surface.

Recently, we considered in [2] another natural family of functionals $\mathcal{F}_c$ which come from the integration of $|J^\perp|^2$. Therefore, it can be used to detect the derivation of a surface from a holomorphic curve. We proved that the equation satisfied by the critical point of the functional (which we called *$\mathcal{F}_c$-critical surface*) is an elliptic system modulo the diffeomorphism group of Σ when $c > 1$. We also proved that any symplectic $\mathcal{F}_c$-critical surface with $c \geq 1$ in a Kähler-Einstein surface with positive scalar curvature is holomorphic.

As in [1], in this note, we will fix the immersion F and the complex structure J_M, and let the Kähler form vary in the fixed Kähler class. We have seen that the situations for $p < 0$ and $p \geq 0$ are different in the definition of the functional. So let us first fix some notations. Let $\mathbb{Z}$ be the set of integer numbers. Denote $\mathbb{Z}^+ := \{p \in \mathbb{Z} : p > 0\}$. Define

$$\mathcal{H} := \{\rho \in C^\infty(M, \mathbb{R}) : \bar{\omega}_\rho = \bar{\omega} + \sqrt{-1}\partial\bar{\partial}\rho \in [\bar{\omega}], \cos\alpha_\rho > 0\}.$$

Here, $[\bar{\omega}]$ is the Kähler class of $\bar{\omega}$, and α_ρ is the Kähler angle of the immersion F with respect to the Kähler form $\bar{\omega}_\rho$ and associated Riemannian metric $\bar{g}_\rho$ defined by (1.1). If $\cos\alpha_0 > 0$, then $\mathcal{H}$ is nonempty open subset of $C^\infty(M, \mathbb{R})$. We can define a functional on $\mathcal{H}$ by

$$L_p(\rho) = \int_\Sigma \cos^p \alpha_\rho d\mu_\rho, \tag{1.4}$$

where $d\mu_\rho$ is the area form of the induced metric $g_\rho = F^*\bar{g}_\rho$ on Σ.

Definition 1.1. *Given a symplectic immersion $F : \Sigma^2 \to (M, \bar{\omega}, J_M, \bar{g})$, we say that **the functional L_p has a critical point** $\rho \in \mathcal{H}$ if for any $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = \rho$*

$$\frac{d}{dt}|_{t=0} L_p(\varphi(t)) = 0.$$

It is easy to see that if the immersion is a J_M-holomorphic curve, then the functional L_p is constant on $\mathcal{H}$ (see Corollary 2.2) and thus it has every point in $\mathcal{H}$ as its critical point. The first result in this paper shows that the existence of a critical point of L_p is enough to guarantee the J_M-holomorphicity:

Corollary 3.2 *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be a symplectic immersion. If for some $p \in \mathbb{Z} - \{1\}$, the functional L_p has a critical point in $\mathcal{H}$, then the immersion is J_M-holomorphic.*

Actually, as in [1], in order to get the desired conclusion, it suffices to check the vanishing of the first variation of L_p just in one specific direction, determined by the distance squared from the submanifold. (See Theorem 3.1.)

For second variation, we have

Definition 1.2. *Given a symplectic immersion $F : \Sigma^2 \to (M, \bar{\omega}, J_M, \bar{g})$, we say that $\rho \in \mathcal{H}_p$ is a **stable point** for the functional L_p if for any $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = \rho$*

$$\frac{d^2}{dt^2}|_{t=0} L_p(\varphi(t)) \geq 0.$$

Note that the definition of L_p-stability does not require ρ to be a critical point of the functional L_p. Our next result shows that the existence of a stable point is also enough to guarantee the J_M-holomorphicity:

Corollary 4.2 *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be a symplectic immersion. If for some $p \in \mathbb{Z} - \{1\}$, the functional L_p has a stable point in $\mathcal{H}$, then the immersion is J_M-holomorphic.*

Notice that in the above statements, we assume that the immersion is symplectic. Actually, for the well-definedness of the functional L_p, this is only needed for $p < 0$. When $p \geq 0$, we can view L_p as a functional defined on the following larger space (which is the one considered in [1]):

$$\tilde{\mathcal{H}} := \{\rho \in C^\infty(M, \mathbb{R}) : \bar{\omega}_\rho = \bar{\omega} + \sqrt{-1}\partial\bar{\partial}\rho \in [\bar{\omega}]\}.$$

We can also define critical point and stable point as above with $\mathcal{H}$ replaced by $\tilde{\mathcal{H}}$.

When $p = 0$, the functional L_0 is exactly the area functional defined on $\tilde{\mathcal{H}}$. In this case, we do not need "symplectic" assumption, and all the above results reduce to [1] for Kähler case.

If $p > 0$ is even, then a Lagrangian immersion achieves the minimum of the L_p and thus also has critical point. In this case, without "symplectic" assumption, we can show that the existence of a critical point or stable point of L_p implies that the immersion is J_M-holomorphic or Lagrangian.

Corollary 1.1. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If for some $p \in 2\mathbb{Z}^+$, the functional L_p has a critical point or stable point in $\tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic or Lagrangian.*

Note that in all the above statements, we need to exclude the case for $p = 1$. Actually, by the definition of Kähler angle (1.2), we have

$$L_1 = \int_\Sigma \cos\alpha d\mu = \int_\Sigma \bar{\omega} = [\bar{\omega}][\Sigma],$$

which is constant in $[\bar{\omega}]$. Thus in this case, for any fixed immersion of Σ in M, L_1 is constant on $\tilde{\mathcal{H}}$, which means that any point in $\tilde{\mathcal{H}}$ is a critical point of L_1. (Compare this with Corollary 2.2, which is true only for J_M-holomorphic immersions.)

2. L_p-FUNCTIONAL FOR A J_M-HOLOMORPHIC CURVE

In this section, we will show that an immersion is J_M-holomorphic if and only if the Kähler angle is zero identically. This a well-known fact. For the purpose of completeness, we give a proof here. As a corollary, we see that the functional L_p is constant on $\mathcal{H}$ if the immersion is J_M-holomorphic.

Let us first recall the definition of J_M-holomorphic immersion:

Definition 2.1. *Let (M^{2n}, J_M) be an almost complex manifold and Σ be a surface. We call an immersion $F : \Sigma \to (M, J_M)$ J_M**-holomorphic** if $(J_M)_{F(p)}$ maps $F_{*p}(T_p\Sigma)$ onto itself for any point $p \in \Sigma$.*

The main observation we will prove in this section is as follows:

Proposition 2.1. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact symplectic manifold with symplectic form $\bar{\omega}$, compatible almost complex structure J_M and associated Riemannian metric $\bar{g}$. Then $F : \Sigma \to (M, J_M)$ is a J_M-holomorphic immersion if and only if $\sin\alpha \equiv 0$.*

Proof: First suppose $F : \Sigma \to (M, J_M)$ is a J_M-holomorphic immersion, then by definition we can easily see that the almost complex structure J_M on M can induce an almost complex structure J_Σ on Σ, such that the immersion $F : (\Sigma, J_\Sigma) \to (M, J_M)$ is (J_Σ, J_M)-holomorphic. That is,

$$J_M \circ F_* = F_* \circ J_\Sigma. \tag{2.1}$$

For any compatible symplectic form $\bar{\omega}$ and associated Riemannian metric $\bar{g}$ defined by (1.1), the induced metric and induced 2-form on Σ are given by

$$g = F^*\bar{g}, \quad \omega = F^*\bar{\omega}.$$

The cosine of the Kähler angle α is define by (1.2):

$$\omega = \cos\alpha d\mu_g.$$

The following facts follow easily from (2.1): for any $X, Y \in T\Sigma$,

(a) $\omega(X, Y) = \bar{\omega}(F_*X, F_*Y) = \bar{g}(J_M F_*X, F_*Y) = \bar{g}(F_*J_\Sigma X, F_*Y) = g(J_\Sigma X, Y)$;

(b) $g(J_\Sigma X, J_\Sigma Y) = \bar{g}(F_*J_\Sigma X, F_*J_\Sigma Y) = \bar{g}(J_M F_*X, J_M F_*Y) = \bar{g}(F_*X, F_*Y) = g(X, Y)$.

Now take any g-orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$ for $p \in \Sigma$. Then by definition and (a),

$$\cos\alpha(p) = \omega(e_1, e_2) = g(J_\Sigma e_1, e_2).$$

By (b), $g(J_\Sigma e_1, e_1) = 0$. Thus, we have

$$J_\Sigma e_1 = g(J_\Sigma e_1, e_2)e_2 = \cos\alpha e_2.$$

Therefore,

$$g(J_\Sigma e_1, J_\Sigma e_1) = \cos^2\alpha(p) g(e_2, e_2) = \cos^2\alpha(p).$$

On the other hand, by (b),

$$g(J_\Sigma e_1, J_\Sigma e_1) = g(e_1, e_1) = 1.$$

Therefore, $\cos^2\alpha(p) = 1$, i.e., $\sin^2\alpha(p) = 0$. As p is arbitrary, we must have $\sin\alpha \equiv 0$ on Σ.

On the contrary, suppose $\sin\alpha \equiv 0$. As $\cos\alpha$ is smooth on Σ, without loss of generality, we can assume that $\cos\alpha \equiv 1$. Take any g-orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$ for $p \in \Sigma$. By (1.2), we have

$$\omega(e_1, e_2) = d\mu_g(e_1, e_2) = 1.$$

On the other hand, by (1.1), we have

$$\omega(e_1, e_2) = \bar{\omega}(F_*e_1, F_*e_2) = \bar{g}(J_M F_*e_1, F_*e_2).$$

Therefore, we have

$$\bar{g}(J_M F_*e_1, F_*e_2) = 1. \tag{2.2}$$

By the choice of $\{e_1, e_2\}$, we get that

$$\bar{g}(F_*e_i, F_*e_j) = g(e_i, e_j) = \delta_{ij}. \tag{2.3}$$

Note that $F_{*p}(T_p\Sigma)$ is a two-plane contained in $T_{F(p)}M$. We denote $N_{F(p)}\Sigma$ the normal space of $F_{*p}(T_p\Sigma)$ in $T_{F(p)}M$. Take a $\bar{g}$-orthonormal basis of $T_{F(p)}M$ $\{F_*e_1, F_*e_2, e_3, \cdots, e_{2n}\}$, such that $N_{F(p)}\Sigma = span\{e_3, \cdots, e_{2n}\}$. Then by (2.2),

$$\begin{aligned} J_M F_*e_1 &= \bar{g}(J_M F_*e_1, F_*e_1)F_*e_1 + \bar{g}(J_M F_*e_1, F_*e_2)F_*e_2 + \sum_{\alpha=3}^{2n} \bar{g}(J_M F_*e_1, e_\alpha)e_\alpha \\ &= \bar{g}(J_M F_*e_1, F_*e_1)F_*e_1 + F_*e_2 + \sum_{\alpha=3}^{2n} \bar{g}(J_M F_*e_1, e_\alpha)e_\alpha. \end{aligned}$$

As by (2.3),

$$\bar{g}(J_M F_*e_1, J_M F_*e_1) = 1 = \bar{g}(F_*e_2, F_*e_2),$$

we must have

$$J_M F_*e_1 = F_*e_2, \tag{2.4}$$

and thus

$$J_M F_*e_2 = -F_*e_1. \tag{2.5}$$

This implies that J_M maps $F_{*p}(T_p\Sigma)$ onto itself. Therefore, $F : \Sigma \to (M, J_M)$ is J_M-holomorphic. Q.E.D.

Corollary 2.2. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact Kähler manifold. If $F : \Sigma \to (M, J_M)$ is a J_M-holomorphic immersion, then L_p-functional is constant on $\mathcal{H}$ for any $p \in \mathbb{Z}$. In particular any $\rho \in \mathcal{H}$ is both a critical point and a stable point for the functional L_p.*

Proof: By Proposition 2.1, for each $\rho \in \mathcal{H}$, $\cos^2\alpha_\rho \equiv 1$ on Σ. As $\cos\alpha_\rho$ is smooth on Σ, without loss of generality, we may assume that $\cos\alpha_\rho \equiv 1$. Then,

$$L_p(\rho) = \int_\Sigma \cos^p \alpha_\rho d\mu_\rho = \int_\Sigma \cos\alpha_\rho d\mu_\rho = \int_\Sigma F^*\bar{\omega}_\rho = [\bar{\omega}_\rho][\Sigma] = [\bar{\omega}][\Sigma],$$

which is independent of $\rho \in \mathcal{H}$. Q.E.D.

3. Critical Point and J_M-holomorphicity

In this and next sections, we will consider the converse of Corollary 2.2. In this section, we compute the first variations of the Kähler angle and L_p-functional when we move the Kähler form on M in the fixed Kähler class. We will prove that if L_p-functional has a critical point, then the immersion is J_M-holomorphic.

Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a compact Kähler manifold. Given $\varphi \in \mathcal{H}$, define

$$\bar{\omega}_\varphi = \bar{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi. \tag{3.1}$$

The associated Kähler metric is given by

$$\bar{g}_\varphi(U, V) = \bar{\omega}_\varphi(U, J_M V). \tag{3.2}$$

Given the immersion $F : \Sigma \to M$, we have the induced metric and 2-form on Σ:

$$g_\varphi = F^*\bar{g}_\varphi, \quad \omega_\varphi = F^*\bar{\omega}_\varphi. \tag{3.3}$$

The cosine of the Kähler angle α_φ is define by

$$\omega_\varphi = \cos\alpha_\varphi d\mu_{g_\varphi}. \tag{3.4}$$

Fix a g-orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$, where $g = g_0$. Then by (1.2),

$$\cos\alpha = \omega(e_1, e_2). \tag{3.5}$$

By (3.4), we have

$$\cos\alpha_\varphi = \frac{\omega_\varphi(e_1, e_2)}{\sqrt{det(g_\varphi(e_i, e_j))}}. \tag{3.6}$$

By (3.1) and (3.3), we have

$$\omega_\varphi = F^*(\bar{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi) = \omega + F^*(\sqrt{-1}\partial\bar{\partial}\varphi), \tag{3.7}$$

so that

$$\omega_\varphi(e_1, e_2) = \cos\alpha + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, F_*e_2). \tag{3.8}$$

On the other hand, by the choice of frame and (3.1), (3.2), (3.3), we have

$$\begin{aligned} g_\varphi(e_i, e_j) &= \bar{g}_\varphi(F_*e_i, F_*e_j) = \bar{\omega}_\varphi(F_*e_i, J_M F_*e_j) \\ &= \bar{\omega}(F_*e_i, J_M F_*e_j) + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_i, J_M F_*e_j) \\ &= \delta_{ij} + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_i, J_M F_*e_j). \end{aligned} \tag{3.9}$$

Namely,

$$(g_\varphi) = \begin{pmatrix} 1 + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, J_M F_*e_1) & (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, J_M F_*e_2) \\ (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_2, J_M F_*e_1) & 1 + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_2, J_M F_*e_2) \end{pmatrix}.$$

Therefore,

$$\begin{aligned} det(g_\varphi) &= 1 + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, J_M F_*e_1) + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_2, J_M F_*e_2) \\ &\quad + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, J_M F_*e_1)(\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_2, J_M F_*e_2) \\ &\quad - [(\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, J_M F_*e_2)]^2. \end{aligned} \tag{3.10}$$

From (3.6) and (3.8), we have

$$\cos\alpha_\varphi = \frac{\cos\alpha + (\sqrt{-1}\partial\bar{\partial}\varphi)(F_*e_1, F_*e_2)}{\sqrt{det(g_\varphi(e_i, e_j))}}. \tag{3.11}$$

It is known that (Lemma 2.1 of [1]), for any $U, V \in TM$, we have

$$(\sqrt{-1}\partial\bar{\partial}\varphi)(U,V) = \frac{1}{2}\left[-(\overline{\nabla}^2\varphi)(U, J_M V) + (\overline{\nabla}^2\varphi)(V, J_M U)\right], \tag{3.12}$$

and thus

$$(\sqrt{-1}\partial\bar{\partial}\varphi)(U, J_M V) = \frac{1}{2}\left[(\overline{\nabla}^2\varphi)(U,V) + (\overline{\nabla}^2\varphi)(J_M U, J_M V)\right]. \tag{3.13}$$

For simplicity, from now on, we will identify e_i with F_*e_i. At a fixed point $p \in \Sigma$, it is easy to see that we can choose a $\bar{g}$-orthonormal frame $\{e_1, e_2, \cdots, e_{2n}\}$ of T_pM, such that $\{e_1, e_2\}$ spans the tangent space of Σ, $\{e_3, \cdots, e_{2n}\}$ spans the normal space of Σ, and the complex structure takes the form

$$J_M = \begin{pmatrix} (J_1)_{4\times4} & 0_{4\times(2n-4)} \\ 0_{(2n-4)\times4} & (J_2)_{(2n-4)\times(2n-4)} \end{pmatrix}, \tag{3.14}$$

where

$$J_1 = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}, \tag{3.15}$$

and J_2 satisfies $J_2^2 = -Id_{2n-4}$.

By (3.12) and (3.14), we have

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}\varphi)(e_1, e_2) &= \frac{1}{2}\left\{-(\overline{\nabla}^2\varphi)(e_1, J_M e_2) + (\overline{\nabla}^2\varphi)(e_2, J_M e_1)\right\} \\ &= \frac{1}{2}\left\{\cos\alpha[(\overline{\nabla}^2\varphi)(e_1,e_1) + (\overline{\nabla}^2\varphi)(e_2,e_2)]\right. \\ &\quad \left. + \sin\alpha[(\overline{\nabla}^2\varphi)(e_1,e_4) + (\overline{\nabla}^2\varphi)(e_2,e_3)]\right\}. \end{aligned} \tag{3.16}$$

By (3.13) and (3.14), we have

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}\varphi)(e_1, J_M e_1) &= \frac{1}{2}\left\{(\overline{\nabla}^2\varphi)(e_1,e_1) + (\overline{\nabla}^2\varphi)(J_M e_1, J_M e_1)\right\} \\ &= \frac{1}{2}\left\{(\overline{\nabla}^2\varphi)(e_1,e_1) + \cos^2\alpha(\overline{\nabla}^2\varphi)(e_2,e_2)\right. \\ &\quad \left. + 2\sin\alpha\cos\alpha(\overline{\nabla}^2\varphi)(e_2,e_3) + \sin^2\alpha(\overline{\nabla}^2\varphi)(e_3,e_3)\right\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}\varphi)(e_2, J_M e_2) &= \frac{1}{2}\left\{(\overline{\nabla}^2\varphi)(e_2,e_2) + (\overline{\nabla}^2\varphi)(J_M e_2, J_M e_2)\right\} \\ &= \frac{1}{2}\left\{(\overline{\nabla}^2\varphi)(e_2,e_2) + \cos^2\alpha(\overline{\nabla}^2\varphi)(e_1,e_1)\right. \\ &\quad \left. + 2\sin\alpha\cos\alpha(\overline{\nabla}^2\varphi)(e_1,e_4) + \sin^2\alpha(\overline{\nabla}^2\varphi)(e_4,e_4)\right\}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}(\sqrt{-1}\partial\bar{\partial}\varphi)(e_1, J_M e_2) &= \frac{1}{2}\left\{(\overline{\nabla}^2\varphi)(e_1,e_2) + (\overline{\nabla}^2\varphi)(J_M e_1, J_M e_2)\right\} \\ &= \frac{1}{2}\left\{\sin^2\alpha[(\overline{\nabla}^2\varphi)(e_1,e_2) - (\overline{\nabla}^2\varphi)(e_3,e_4)]\right. \\ &\quad \left. -\sin\alpha\cos\alpha[(\overline{\nabla}^2\varphi)(e_1,e_3) + (\overline{\nabla}^2\varphi)(e_2,e_4)]\right\}. \end{aligned} \tag{3.19}$$

We will compute the first variation of the Kähler angle. Let $\varphi(t)$ be a family of Kähler potential on M so that $\varphi(0)\equiv 0$ and $\dot{\varphi} = \psi$. Then by (3.10), we have

$$\frac{d}{dt}|_{t=0} det(g_{\varphi(t)}) = (\sqrt{-1}\partial\bar{\partial}\psi)(e_1, J_M e_1) + (\sqrt{-1}\partial\bar{\partial}\psi)(e_2, J_M e_2). \tag{3.20}$$

By (3.11) and using (3.16), (3.17), (3.18) and (3.20), we have

$$\begin{aligned} &\frac{d}{dt}|_{t=0}\cos\alpha_{\varphi(t)} \\ =\ & (\sqrt{-1}\partial\bar{\partial}\psi)(e_1,e_2) - \frac{1}{2}\cos\alpha\frac{d}{dt}|_{t=0} det(g_{\varphi(t)}) \\ =\ & (\sqrt{-1}\partial\bar{\partial}\psi)(e_1,e_2) - \frac{1}{2}\cos\alpha[(\sqrt{-1}\partial\bar{\partial}\psi)(e_1, J_M e_1) + (\sqrt{-1}\partial\bar{\partial}\psi)(e_2, J_M e_2)] \\ =\ & \frac{1}{2}\left\{\cos\alpha[(\overline{\nabla}^2\psi)(e_1,e_1) + (\overline{\nabla}^2\psi)(e_2,e_2)] + \sin\alpha[(\overline{\nabla}^2\psi)(e_1,e_4) + (\overline{\nabla}^2\psi)(e_2,e_3)]\right\} \\ & -\frac{1}{4}\cos\alpha\left\{(1+\cos^2\alpha)(\overline{\nabla}^2\psi)(e_1,e_1) + (1+\cos^2\alpha)(\overline{\nabla}^2\psi)(e_2,e_2)\right. \\ & \qquad +2\sin\alpha\cos\alpha(\overline{\nabla}^2\psi)(e_2,e_3) + \sin^2\alpha(\overline{\nabla}^2\psi)(e_3,e_3) \\ & \qquad \left. +2\sin\alpha\cos\alpha(\overline{\nabla}^2\psi)(e_1,e_4) + \sin^2\alpha(\overline{\nabla}^2\psi)(e_4,e_4)\right\} \\ =\ & \frac{1}{4}\sin^2\alpha\left\{\cos\alpha\left[(\overline{\nabla}^2\psi)(e_1,e_1) + (\overline{\nabla}^2\psi)(e_2,e_2) - (\overline{\nabla}^2\psi)(e_3,e_3) - (\overline{\nabla}^2\psi)(e_4,e_4)\right]\right. \\ & \qquad \left. +2\sin\alpha\left[(\overline{\nabla}^2\psi)(e_1,e_4) + (\overline{\nabla}^2\psi)(e_2,e_3)\right]\right\}. \end{aligned} \tag{3.21}$$

Now we can prove the first theorem in this section:

Theorem 3.1. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F: \Sigma^2 \to M$ be a symplectic immersion. Set $d\colon M \to \boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q) = dist(Q, F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d}{dt}|_{t=0} L_p(\bar{\omega}_\rho + t\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{2})) = 0$$

for some $p \in \mathbb{Z} - \{1\}$ and $\rho \in \mathcal{H}$, then the immersion is J_M-holomorphic.

Proof: To simplify the notation, by assumption, without loss of generality, we assume that $\rho \equiv 0$ so that $\bar{\omega}_\rho = \bar{\omega}$. Let $\varphi(t)$ be any curve in $\mathcal{H}$ so that $\varphi(0)\equiv 0$ and $\dot{\varphi} = \psi$. At a fixed point $p \in \Sigma$, take an orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$ so that the complex structure

J_M takes the form (3.14). By (3.9), it is easy to see that

$$\frac{d}{dt}|_{t=0}d\mu_t = \frac{1}{2}\sum_{i=1}^{2}(\sqrt{-1}\partial\bar{\partial}\psi)(e_i, J_M e_i)d\mu. \tag{3.22}$$

By direct computation using (3.17) and (3.18), we obtain that

$$\begin{aligned}\frac{d}{dt}|_{t=0}d\mu_t &= \frac{1}{4}\Big\{(1+\cos^2\alpha)(\overline{\nabla}^2\psi)(e_1,e_1) + (1+\cos^2\alpha)(\overline{\nabla}^2\psi)(e_2,e_2)\\ &\quad +2\sin\alpha\cos\alpha(\overline{\nabla}^2\psi)(e_2,e_3) + \sin^2\alpha(\overline{\nabla}^2\psi)(e_3,e_3)\\ &\quad +2\sin\alpha\cos\alpha(\overline{\nabla}^2\psi)(e_1,e_4) + \sin^2\alpha(\overline{\nabla}^2\psi)(e_4,e_4)\Big\}.\end{aligned} \tag{3.23}$$

By the definition of the functional L_p, (3.21) and (3.23), we compute

$$\begin{aligned}&\frac{d}{dt}|_{t=0}L_p(\varphi(t))\\ =\ & p\int_\Sigma \cos^{p-1}\alpha\frac{d}{dt}|_{t=0}\cos\alpha_{\varphi(t)}d\mu + \int_\Sigma\cos^p\alpha\frac{d}{dt}|_{t=0}d\mu_t\\ =\ & \int_\Sigma \frac{p\cos^{p-1}\alpha\sin^2\alpha}{4}\Big\{\cos\alpha\Big[(\overline{\nabla}^2\psi)(e_1,e_1)+(\overline{\nabla}^2\psi)(e_2,e_2)\\ &\quad -(\overline{\nabla}^2\psi)(e_3,e_3)-(\overline{\nabla}^2\psi)(e_4,e_4)\Big]\\ &\quad +2\sin\alpha\Big[(\overline{\nabla}^2\psi)(e_1,e_4)+(\overline{\nabla}^2\psi)(e_2,e_3)\Big]\Big\}d\mu\\ &+\int_\Sigma\frac{\cos^p\alpha}{4}\Big\{(1+\cos^2\alpha)\Big[(\overline{\nabla}^2\psi)(e_1,e_1)+(\overline{\nabla}^2\psi)(e_2,e_2)\Big]\\ &\quad +\sin^2\alpha\Big[(\overline{\nabla}^2\psi)(e_3,e_3)+(\overline{\nabla}^2\psi)(e_4,e_4)\Big]\\ &\quad +2\sin\alpha\cos\alpha\Big[(\overline{\nabla}^2\psi)(e_2,e_3)+(\overline{\nabla}^2\psi)(e_1,e_4)\Big]\Big\}d\mu\\ =\ & \int_\Sigma\Big\{\frac{1}{4}\cos^p\alpha(1+\cos^2\alpha+p\sin^2\alpha)\Big[(\overline{\nabla}^2\psi)(e_1,e_1)+(\overline{\nabla}^2\psi)(e_2,e_2)\Big]\\ &\quad +\frac{1-p}{4}\sin^2\alpha\cos^p\alpha\Big[(\overline{\nabla}^2\psi)(e_3,e_3)+(\overline{\nabla}^2\psi)(e_4,e_4)\Big]\\ &\quad +\frac{1}{2}\sin\alpha\cos^{p-1}\alpha(\cos^2\alpha+p\sin^2\alpha)\Big[(\overline{\nabla}^2\psi)(e_2,e_3)+(\overline{\nabla}^2\psi)(e_1,e_4)\Big]\Big\}d\mu.\end{aligned} \tag{3.24}$$

Next, we will take special test function ψ. Actually, we will take the same ψ as in the proof of Theorem 2.4 of [1]. By Corollary 2.2, to prove the theorem, it suffices to show that $\sin\alpha\equiv 0$ on Σ. We identify Σ with its image in M. Denote d the distance function of M from Σ with respect to the metric $\bar{g}$. Namely, for $Q\in M$, $d(Q)=dist_{\bar{g}}(Q,\Sigma)$. Then it is known that $\xi=\frac{1}{2}d^2$ is smooth in a neighborhood of Σ in M, and for any $x_0\in\Sigma$, $Hess(\xi)(x_0)$ represents the orthogonal projection on the normal space to Σ at x_0. (See Proposition 2.5 of [1].) Namely, for each $U,V\in T_{x_0}M$ and $x_0\in\Sigma$, we have

$$(\overline{\nabla}^2\xi)(U,V)(x_0)=\langle U^\perp, V^\perp\rangle, \tag{3.25}$$

where $T_{x_0}M=T_{x_0}\Sigma\oplus N_{x_0}\Sigma$ and $U^\perp$ is the projection of U onto $N_{x_0}\Sigma$. We take ψ to be a smooth function on M such that $\psi=\xi=\frac{1}{2}d^2$ in a neighborhood of Σ in M. Then by

(3.25), for $\bar{\omega}(t)=\bar{\omega}+t\sqrt{-1}\partial\bar{\partial}\psi$, we have from (3.24) that

$$\frac{d}{dt}|_{t=0}L(\bar{\omega}+t\sqrt{-1}\partial\bar{\partial}\psi)=\frac{1-p}{2}\int_{\Sigma}\sin^2\alpha\cos^p\alpha d\mu. \tag{3.26}$$

By our assumption, $\cos\alpha>0$, $p\neq 1$ and $\frac{d}{dt}|_{t=0}L(\bar{\omega}+t\sqrt{-1}\partial\bar{\partial}\psi)=0$. Therefore, we must have $\sin\alpha\equiv 0$. This proves the theorem. Q.E.D.

Corollary 3.2. *Let $(M^{2n},\bar{\omega},J_M,\bar{g})$ be a Kähler manifold as above and $F:\Sigma^2\to M$ be a symplectic immersion. If for some $p\in\mathbb{Z}-\{1\}$, the functional L_p has a critical point in $\mathcal{H}$, then the immersion is J_M-holomorphic.*

When $p=0$, the integrand of the right hand side of (3.26) becomes $\frac{1}{2}\sin^2\alpha$. In this case, we do not need the immersion to be "symplectic", and the theorem reduces to Theorem 1.2 of [1].

Theorem 3.3. *([1]) Let $(M^{2n},\bar{\omega},J_M,\bar{g})$ be a Kähler manifold as above and $F:\Sigma^2\to M$ be an immersion. Set $d\colon M\to\boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q)=dist(Q,F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d}{dt}|_{t=0}L_0(\bar{\omega}_\rho+t\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{2}))=0,$$

then the immersion is J_M-holomorphic.

Corollary 3.4. *([1]) Let $(M^{2n},\bar{\omega},J_M,\bar{g})$ be a Kähler manifold as above and $F:\Sigma^2\to M$ be an immersion. If the functional L_0 has a critical point in $\mathcal{H}_0$, then the immersion is J_M-holomorphic.*

When $p>0$, we can say more. In this case, we allow $\cos\alpha$ to have zeros in the definition of the functional L_p. Obviously, a Lagrangian immersion (i.e., $\cos\alpha\equiv 0$) achieves the minimum of the functional L_p while a holomorphic immersion achieves the maximum when p is even. Therefore, in this case, both Lagrangian immersion and holomorphic immersion have critical points. We will sow that the converse is also true.

Theorem 3.5. *Let $(M^{2n},\bar{\omega},J_M,\bar{g})$ be a Kähler manifold as above and $F:\Sigma^2\to M$ be an immersion. Set $d\colon M\to\boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q)=dist(Q,F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d}{dt}|_{t=0}L_p(\bar{\omega}_\rho+t\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{2}))=0$$

for some $p\in 2\mathbb{Z}^+$ and $\rho\in\tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic or Lagrangian.

Proof: Proceeding in the same way as in the proof of Theorem 3.1, we finally obtain (3.26). As p is a positive even integer, by the assumption, we must have

$$\sin^2\alpha\cos^p\alpha\equiv 0,\quad on\ \Sigma.$$

Denote

$$\Omega_1=\{x\in\Sigma:\sin\alpha(x)=0\},\quad \Omega_2=\{x\in\Sigma:\cos\alpha(x)=0\}.$$

Then we have that: $\Sigma=\Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2=\emptyset$ and both Ω_1 and Ω_2 are closed set. As Σ is connected, we must have $\Omega_1=\emptyset$ or $\Omega_1=\Sigma$.

If $\Omega_1=\emptyset$, then $\Omega_2=\Sigma$, i.e., $\cos\alpha\equiv 0$ on Σ. In this case, the immersion is Lagrangian.

If $\Omega_1 = \Sigma$, i.e., $\sin\alpha \equiv 0$ on Σ, then by Proposition 2.1, the immersion is J_M-holomorphic. Q.E.D.

Corollary 3.6. *Let $(M^{2n}, \bar\omega, J_M, \bar g)$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If for some $p \in 2\mathbb{Z}^+$, the functional L_p has a critical point in $\tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic or Lagrangian.*

4. Stable Point and J_M-holomorphicity

In this section, we will compute the second variation formula for the functional L_p and prove that existence of stable point implies J_M-holomorphicity. Instead of computing the variations of $\cos\alpha$ and $d\mu$ separately, we will compute the variations by combining them together.

Let $(M, \bar\omega, J_M, \bar g)$ be as in the previous section and $F : \Sigma \to M$ be an immersion. (As before, we assume F to be symplectic if $p < 0$.) At a fixed point, we can choose the local coordinate $\{x_1, x_2\}$ on Σ such that at that point, $\{\partial_{x_1}, \partial_{x_2}\}$ is g-orthonormal. Using (3.11), we can rewrite L as

$$L_p(\varphi) = \int_\Sigma (\cos\alpha + (\sqrt{-1}\partial\bar\partial\varphi)(e_1, e_2))^p det(g_\varphi)^{\frac{1-p}{2}} dx_1 \wedge dx_2. \tag{4.1}$$

Now take any curve $\varphi(t)$ in $\mathcal{H}$ with $\varphi(0) \equiv 0$, $\dot\varphi = \psi$ and $\ddot\varphi = \eta$. Denote

$$\nu_p(t) = (\cos\alpha + (\sqrt{-1}\partial\bar\partial\varphi)(e_1, e_2))^p det(g_\varphi)^{\frac{1-p}{2}}. \tag{4.2}$$

Then, we have

$$\begin{aligned}\frac{d}{dt}\nu_p(t) &= p(\cos\alpha + (\sqrt{-1}\partial\bar\partial\varphi)(e_1, e_2))^{p-1} det(g_\varphi)^{\frac{1-p}{2}} \frac{d}{dt}(\sqrt{-1}\partial\bar\partial\varphi(t))(e_1, e_2)\\ &\quad + \frac{1-p}{2}(\cos\alpha + (\sqrt{-1}\partial\bar\partial\varphi)(e_1, e_2))^p det(g_\varphi)^{\frac{-1-p}{2}} \frac{d}{dt} det(g_{\varphi(t)}).\end{aligned}$$

Hence, by using (3.10) and (3.20), we have

$$\begin{aligned}&\frac{d^2}{dt^2}|_{t=0}\nu_p(t)\\ =\ & p\cos^{p-1}\alpha \frac{d^2}{dt^2}|_{t=0}(\sqrt{-1}\partial\bar\partial\varphi(t))(e_1, e_2)\\ &+p(1-p)\cos^{p-1}\alpha \frac{d}{dt}|_{t=0}(\sqrt{-1}\partial\bar\partial\varphi(t))(e_1, e_2)\frac{d}{dt}|_{t=0} det(g_{\varphi(t)})\\ &+p(p-1)\cos^{p-2}\alpha\left[\frac{d}{dt}|_{t=0}(\sqrt{-1}\partial\bar\partial\varphi(t))(e_1, e_2)\right]^2\\ &+\frac{(p-1)(p+1)}{4}\cos^p\alpha\left[\frac{d}{dt}|_{t=0} det(g_{\varphi(t)})\right]^2 + \frac{1-p}{2}\cos^p\alpha \frac{d^2}{dt^2}|_{t=0} det(g_{\varphi(t)})\\ =\ & p\cos^{p-1}\alpha(\sqrt{-1}\partial\bar\partial\eta)(e_1, e_2)\\ &+p(1-p)\cos^{p-1}\alpha(\sqrt{-1}\partial\bar\partial\psi)(e_1, e_2)[(\sqrt{-1}\partial\bar\partial\psi)(e_1, J_Me_1) + (\sqrt{-1}\partial\bar\partial\psi)(e_2, J_Me_2)]\\ &+p(p-1)\cos^{p-2}\alpha\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1, e_2)\right]^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{(p-1)(p+1)}{4}\cos^p\alpha\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)\right]^2\\
&+\frac{1-p}{2}\cos^p\alpha\left\{(\sqrt{-1}\partial\bar\partial\eta)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\eta)(e_2,J_Me_2)\right.\\
&\qquad\left.+2(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)-2\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_2)\right]^2\right\}\\
= &\cos^p\alpha\left\{\frac{1-p}{2}\left[(\sqrt{-1}\partial\bar\partial\eta)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\eta)(e_2,J_Me_2)\right]+\frac{p(\sqrt{-1}\partial\bar\partial\eta)(e_1,e_2)}{\cos\alpha}\right\}\\
&+p(1-p)\cos^{p-1}\alpha(\sqrt{-1}\partial\bar\partial\psi)(e_1,e_2)[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)]\\
&+p(p-1)\cos^{p-2}\alpha\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1,e_2)\right]^2\\
&+\frac{(p-1)(p+1)}{4}\cos^p\alpha\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)\right]^2\\
&+(1-p)\cos^p\alpha\left\{(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)-[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_2)]^2\right\}.
\end{aligned}
$$

Therefore, the second variation formula for the functional L_p is given by

$$
\begin{aligned}
&\frac{d^2}{dt^2}|_{t=0}L_p(\varphi(t))\\
= &\int_\Sigma\cos^p\alpha\left\{\frac{1-p}{2}\left[(\sqrt{-1}\partial\bar\partial\eta)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\eta)(e_2,J_Me_2)\right]+\frac{p(\sqrt{-1}\partial\bar\partial\eta)(e_1,e_2)}{\cos\alpha}\right\}d\mu\\
&+p(1-p)\int_\Sigma\cos^{p-1}\alpha(\sqrt{-1}\partial\bar\partial\psi)(e_1,e_2)[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)]d\mu\\
&+p(p-1)\int_\Sigma\cos^{p-2}\alpha\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1,e_2)\right]^2d\mu\\
&+\frac{(p-1)(p+1)}{4}\int_\Sigma\cos^p\alpha\left[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)+(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)\right]^2d\mu\\
&+(1-p)\int_\Sigma\cos^p\alpha\left\{(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_1)(\sqrt{-1}\partial\bar\partial\psi)(e_2,J_Me_2)-[(\sqrt{-1}\partial\bar\partial\psi)(e_1,J_Me_2)]^2\right\}d\mu.
\end{aligned}
\tag{4.3}
$$

Our main result in this section is as follows:

Theorem 4.1. *Let $(M^{2n},\bar\omega,J_M,\bar g)$ be a Kähler manifold as above and $F:\Sigma^2\to M$ be a symplectic immersion. Set $d\colon M\to\boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q)=dist(Q,F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d^2}{dt^2}|_{t=0}L_p(\bar\omega_\rho+t^2\sqrt{-1}\partial\bar\partial(\frac{d^2}{4}))=0$$

for some $p\in\mathbb{Z}-\{1\}$ and $\rho\in\mathcal{H}$, then the immersion is J_M-holomorphic.

Proof: To simplify the notation, by assumption, without loss of generality, we assume that $\rho\equiv0$ so that $\bar\omega_\rho=\bar\omega$. As before, we will take special variations to prove that $\sin\alpha\equiv0$. We take $\varphi(t)=\frac{t^2}{2}\eta$ so that $\psi\equiv0$. Then the second variation formula (4.3) becomes

$$\frac{d^2}{dt^2}|_{t=0}L_p(\varphi(t))$$

$$= \int_\Sigma \cos^p\alpha \left\{ \frac{1-p}{2}\left[(\sqrt{-1}\partial\bar\partial\eta)(e_1, J_Me_1) + (\sqrt{-1}\partial\bar\partial\eta)(e_2, J_Me_2)\right] + \frac{p(\sqrt{-1}\partial\bar\partial\eta)(e_1,e_2)}{\cos\alpha}\right\} d\mu. \tag{4.4}$$

Now, we take η to be a smooth function on M so that $\eta = \xi = \frac{d^2}{2}$ in a neighborhood of Σ. Here ξ is the function appearing in the proof of Theorem 3.1. By (3.25) and the expressions (3.16), (3.17) and (3.18), we see that when restricting on Σ, we have

$$(\sqrt{-1}\partial\bar\partial\eta)(e_1,e_2) \equiv 0,$$

and

$$(\sqrt{-1}\partial\bar\partial\eta)(e_1, J_Me_1) = (\sqrt{-1}\partial\bar\partial\eta)(e_2, J_Me_2) = \frac{1}{2}\sin^2\alpha.$$

Therefore, for this choice of η, we have

$$\frac{d^2}{dt^2}|_{t=0} L_p(\bar\omega_\rho + t^2\sqrt{-1}\partial\bar\partial(\frac{d^2}{4})) = \frac{1-p}{2}\int_\Sigma \sin^2\alpha\cos^p\alpha d\mu.$$

By our assumption, $\cos\alpha > 0$, $p \neq 1$ and $\frac{d}{dt}|_{t=0}L(\bar\omega_\rho + t^2\sqrt{-1}\partial\bar\partial(\frac{d^2}{4})) = 0$. Therefore, we must have $\sin\alpha \equiv 0$. This proves the theorem. Q.E.D.

Corollary 4.2. *Let $(M^{2n}, \bar\omega, J_M, \bar g)$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be a symplectic immersion. If the functional L_p ($p \in \mathbb{Z} - \{1\}$) has a stable point in $\mathcal{H}$, then the immersion is J_M-holomorphic.*

Proof: As before without loss of generality, we can assume that L_p has a stable point $\rho \equiv 0$. Then by Definition 1.2, we see that for any $\eta \in C^\infty(M, \mathbb{R})$, we have

$$\frac{d^2}{dt^2}|_{t=0} L_p(\bar\omega + \frac{t^2}{2}\sqrt{-1}\partial\bar\partial\eta) \geq 0. \tag{4.5}$$

By (4.4), we can see that

$$\frac{d^2}{dt^2}|_{t=0} L_p(\bar\omega - \frac{t^2}{2}\sqrt{-1}\partial\bar\partial\eta) = -\frac{d^2}{dt^2}|_{t=0} L_p(\bar\omega + \frac{t^2}{2}\sqrt{-1}\partial\bar\partial\eta).$$

Replacing η by $-\eta$ in (4.5), we can obtain that

$$\frac{d^2}{dt^2}|_{t=0} L_p(\bar\omega + \frac{t^2}{2}\sqrt{-1}\partial\bar\partial\eta) = 0, \tag{4.6}$$

for any $\eta \in C^\infty(M, \mathbb{R})$. Now the Corollary follows immediately form Theorem 4.1. Q.E.D.

As in Section 3, we can remove the "symplectic" assumption for $p = 0$ ([1]):

Theorem 4.3. *Let $(M^{2n}, \bar\omega, J_M, \bar g)$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. Set $d\colon M \to \boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q) = dist(Q, F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d^2}{dt^2}|_{t=0} L_0(\bar\omega_\rho + t^2\sqrt{-1}\partial\bar\partial(\frac{d^2}{4})) = 0$$

for some $\rho \in \tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic.

Corollary 4.4. *Let $(M^{2n}, \bar\omega, J_M, \bar g)$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If the functional L_0 has a stable point in $\tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic.*

When $p > 0$, arguing in the same way as in the proof of Theorem 3.5, we can obtain:

Theorem 4.5. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. Set $d\colon M \to \boldsymbol{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to M of the distance function from $F(\Sigma)$, i.e. $d(Q) = dist(Q, F(\Sigma))$ for Q sufficiently near $F(\Sigma)$. If*

$$\frac{d^2}{dt^2}|_{t=0} L_p(\bar{\omega}_\rho + t^2\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{4})) = 0$$

for some $p \in 2\mathbb{Z}^+$ and $\rho \in \tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic or Lagrangian.

Corollary 4.6. *Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If for some $p \in 2\mathbb{Z}^+$, the functional L_p has a stable point in $\tilde{\mathcal{H}}$, then the immersion is J_M-holomorphic or Lagrangian.*

References

[1] C. Arezzo and J. Sun, *A variational characterization of J-holomorphic curves in symplectic manifolds*, arXiv:1211.7016, accepted by Crelle's Journal.

[2] C. Arezzo and J. Sun, *A variational characterization of complex manifolds*, preprint.

[3] S. S. Chern and J. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math., **105** (1983), 59-83.

[4] X. Han and J. Li, *Symplectic critical surfaces in Kähler surfaces*, J. Eur. Math. Soc. (JEMS), **12** (2010), no. 2, 505-527.

[5] X. Han and J. Li, *The second variation of the functional L of symplectic critical surfaces in Kähler surfaces*, preprint.

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