Abstract. In this note, we give another variational characterization of $J_M$-holomorphic curves in Kähler manifolds, generalizing our results in [1]. We prove that if the functional $L_p$ has a critical point or a stable point in the fixed Kähler class, then the immersion is $J_M$-holomorphic or Lagrangian.

Mathematics Subject Classification (2010): 53A10 (primary), 53D05 (secondary).

1. Introduction

Let $(M^{2n}, \tilde{\omega}, J_M, \tilde{g})$ be a compact Kähler manifold of complex dimension $n$ with Kähler form $\tilde{\omega}$, compatible complex structure $J_M$, and associated Riemannian metric $\tilde{g}$. Namely, for $U, V \in TM$,

$$\tilde{g}(U, V) = \tilde{\omega}(U, J_M V).$$

Let $\Sigma$ be a closed connected real surface and $F : \Sigma \to M$ be an immersion. When an immersion is $J_M$-holomorphic is an interesting question in differential geometry. Recently, we ([1]) gave a variational characterization of $J_M$-holomorphic curves in symplectic manifolds. More precisely, we consider the change of the area functional according to the change of the symplectic form on $M$ in the fixed cohomology class (with fixed immersion $F$ and fixed almost complex structure $J_M$ on $M$). Our first result (Theorem 2.4 of [1]) says that if the area functional has a critical point, then the immersion is $J_M$-holomorphic. For the stable case, our second theorem (Theorem 3.2 of [1]) says that, if the area functional has a compatible stable point, then the immersion is also $J_M$-holomorphic. The area functional is a natural candidate to be considered because for a $J_M$-holomorphic immersion, the area functional is constant in the same cohomology class (Proposition 2.2 of [1]). Thus every point is both a critical point and stable point for a $J_M$-holomorphic curve. Recently, we generalized the results in [1] to arbitrary dimension and codimension as well as current case ([2]).

In this note, we consider a family of more general functionals defined in terms of the Kähler angle. Let $(M^{2n}, \tilde{\omega}, J_M, \tilde{g})$ be a compact Kähler manifold as above. Recall that the Kähler angle $\alpha$ of a surface $\Sigma^2$ in $M$ is defined by ([3])

$$\tilde{\omega}|\Sigma = \cos \alpha d\mu_\Sigma,$$

where $d\mu_\Sigma$ is the induced volume form on $\Sigma$. We call an immersion $F : \Sigma \to M$ symplectic if $\cos \alpha > 0$ and Lagrangian if $\cos \alpha \equiv 0$. For a given immersion $F : \Sigma \to (M^{2n}, \tilde{\omega}, J_M, \tilde{g})$,
we can define the following functional

\[(1.3)\quad L_p = \int_{\Sigma} \cos^p \alpha \mu_g, \quad p \in \mathbb{Z}.\]

When \( p < 0 \), we assume the immersion to be symplectic in order to guarantee that the integral makes sense. When \( p = 0 \), \( L_0 \) is just the area functional.

When \( p = -1 \), by fixing the Kähler metric on \( M \) and moving the surface, Han-Li ([4]) computed the Euler-Lagrangian equation for the critical point of the functional \( L_{-1} \), which they called \textit{symplectic critical surface}. They also studied some properties of symplectic critical surface. For example, similar to symplectic minimal surfaces, they proved that if \( M \) is a Kähler-Einstein surface with nonnegative scalar curvature, then every symplectic critical surface is holomorphic. Later on, Han-Li ([5]) computed the second variation formula for the functional \( L_{-1} \), and proved some rigidity results on stable symplectic critical surface.

Recently, we considered in [2] another natural family of functionals \( \mathcal{F}_c \) which come from the integration of \( |J^\perp|^2 \). Therefore, it can be used to detect the derivation of a surface from a holomorphic curve. We proved that the equation satisfied by the critical point of the functional \( L_{-1} \), which they called \textit{symplectic critical surface}. They also studied some properties of symplectic critical surface. For example, similar to symplectic minimal surfaces, they proved that if \( M \) is a Kähler-Einstein surface with nonnegative scalar curvature, then every symplectic critical surface is holomorphic. Later on, Han-Li ([5]) computed the second variation formula for the functional \( L_{-1} \), and proved some rigidity results on stable symplectic critical surface.

As in [1], in this note, we will fix the immersion \( F \) and the complex structure \( J_M \), and let the Kähler form vary in the fixed Kähler class. We have seen that the situations for \( p < 0 \) and \( p \geq 0 \) are different in the definition of the functional. So let us first fix some notations. Let \( \mathbb{Z} \) be the set of integer numbers. Denote \( \mathbb{Z}^+ := \{ p \in \mathbb{Z} : p > 0 \} \). Define

\[\mathcal{H} := \{ \rho \in C^\infty(M, \mathbb{R}) : \bar{\omega}_\rho = \bar{\omega} + \sqrt{-1} \partial \bar{\partial} \rho \in [\bar{\omega}], \cos \alpha_\rho > 0 \}.\]

Here, \([\bar{\omega}]\) is the Kähler class of \( \bar{\omega} \), and \( \alpha_\rho \) is the Kähler angle of the immersion \( F \) with respect to the Kähler form \( \bar{\omega}_\rho \) and associated Riemannian metric \( \bar{g}_\rho \) defined by (1.1). If \( \cos \alpha_0 > 0 \), then \( \mathcal{H} \) is nonempty open subset of \( C^\infty(M, \mathbb{R}) \). We can define a functional on \( \mathcal{H} \) by

\[(1.4)\quad L_p(\rho) = \int_{\Sigma} \cos^p \alpha_\rho d\mu_\rho,\]

where \( d\mu_\rho \) is the area form of the induced metric \( g_\rho = F^* \bar{g}_\rho \) on \( \Sigma \).

\textbf{Definition 1.1.} Given a symplectic immersion \( F : \Sigma^2 \to (M, \bar{\omega}, J_M, \bar{g}) \), we say that the functional \( L_p \) has a critical point \( \rho \in \mathcal{H} \) if for any \( \varphi(t) \in \mathcal{H} \) with \( \varphi(0) = \rho \)

\[\frac{d}{dt}|_{t=0} L_p(\varphi(t)) = 0.\]

It is easy to see that if the immersion is a \( J_M \)-holomorphic curve, then the functional \( L_p \) is constant on \( \mathcal{H} \) (see Corollary 2.2) and thus it has every point in \( \mathcal{H} \) as its critical point. The first result in this paper shows that the existence of a critical point of \( L_p \) is enough to guarantee the \( J_M \)-holomorphicity:

\textbf{Corollary 3.2} Let \((M^{2n}, \bar{\omega}, J_M, \bar{g})\) be a Kähler manifold as above and \( F : \Sigma^2 \to M \) be a symplectic immersion. If for some \( p \in \mathbb{Z} \setminus \{-1\} \), the functional \( L_p \) has a critical point in \( \mathcal{H} \), then the immersion is \( J_M \)-holomorphic.
Actually, as in [1], in order to get the desired conclusion, it suffices to check the vanishing of the first variation of $L_p$ just in one specific direction, determined by the distance squared from the submanifold. (See Theorem 3.1.)

For second variation, we have

**Definition 1.2.** Given a symplectic immersion $F : \Sigma^2 \to (M, \bar{\omega}, J_M, \bar{g})$, we say that $\rho \in \mathcal{H}_p$ is a **stable point** for the functional $L_p$ if for any $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = \rho$

\[
\frac{d^2}{dt^2}|_{t=0} L_p(\varphi(t)) \geq 0.
\]

Note that the definition of $L_p$-stability does not require $\rho$ to be a critical point of the functional $L_p$. Our next result shows that the existence of a stable point is also enough to guarantee the $J_M$-holomorphicity:

**Corollary 4.2** Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be a symplectic immersion. If for some $p \in \mathbb{Z} - \{1\}$, the functional $L_p$ has a stable point in $\mathcal{H}$, then the immersion is $J_M$-holomorphic.

Notice that in the above statements, we assume that the immersion is symplectic. Actually, for the well-definedness of the functional $L_p$, this is only needed for $p < 0$. When $p \geq 0$, we can view $L_p$ as a functional defined on the following larger space (which is the one considered in [1]):

\[
\tilde{\mathcal{H}} := \{ \rho \in C^\infty(M, \mathbb{R}) : \bar{\omega}_\rho = \bar{\omega} + \sqrt{-1} \partial \bar{\partial} \rho \in [\bar{\omega}] \}.
\]

We can also define critical point and stable point as above with $\mathcal{H}$ replaced by $\tilde{\mathcal{H}}$.

When $p = 0$, the functional $L_0$ is exactly the area functional defined on $\tilde{\mathcal{H}}$. In this case, we do not need "symplectic" assumption, and all the above results reduce to [1] for Kähler case.

If $p > 0$ is even, then a Lagrangian immersion achieves the minimum of the $L_p$ and thus also has critical point. In this case, without "symplectic" assumption, we can show that the existence of a critical point or stable point of $L_p$ implies that the immersion is $J_M$-holomorphic or Lagrangian.

**Corollary 1.1.** Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If for some $p \in 2\mathbb{Z}^+$, the functional $L_p$ has a critical point or stable point in $\tilde{\mathcal{H}}$, then the immersion is $J_M$-holomorphic or Lagrangian.

Note that in all the above statements, we need to exclude the case for $p = 1$. Actually, by the definition of Kähler angle (1.2), we have

\[
L_1 = \int_\Sigma \cos \alpha d\mu = \int_\Sigma \bar{\omega} = [\bar{\omega}][\Sigma],
\]

which is constant in $[\bar{\omega}]$. Thus in this case, for any fixed immersion of $\Sigma$ in $M$, $L_1$ is constant on $\tilde{\mathcal{H}}$, which means that any point in $\tilde{\mathcal{H}}$ is a critical point of $L_1$. (Compare this with Corollary 2.2, which is true only for $J_M$-holomorphic immersions.)
2. \(L_p\)-FUNCTIONAL FOR A \(J_M\)-HOLomorphic CURVE

In this section, we will show that an immersion is \(J_M\)-holomorphic if and only if the Kähler angle is zero identically. This a well-known fact. For the purpose of completeness, we give a proof here. As a corollary, we see that the functional \(L_p\) is constant on \(\mathcal{H}\) if the immersion is \(J_M\)-holomorphic.

Let us first recall the definition of \(J_M\)-holomorphic immersion:

**Definition 2.1.** Let \((M^{2n}, J_M)\) be an almost complex manifold and \(\Sigma\) be a surface. We call an immersion \(F: \Sigma \to (M, J_M)\) \(J_M\)-holomorphic if \((J_M)_F(p)\) maps \(F^*(T_p \Sigma)\) onto itself for any point \(p \in \Sigma\).

The main observation we will prove in this section is as follows:

**Proposition 2.1.** Let \((M^{2n}, \bar{\omega}, J_M, \bar{g})\) be a compact symplectic manifold with symplectic form \(\bar{\omega}\), compatible almost complex structure \(J_M\) and associated Riemannian metric \(\bar{g}\). Then \(F: \Sigma \to (M, J_M)\) is a \(J_M\)-holomorphic immersion if and only if \(\sin \alpha \equiv 0\).

**Proof:** First suppose \(F: \Sigma \to (M, J_M)\) is a \(J_M\)-holomorphic immersion, then by definition we can easily see that the almost complex structure \(J_M\) on \(M\) can induce an almost complex structure \(J_{\Sigma}\) on \(\Sigma\), such that the immersion \(F: (\Sigma, J_{\Sigma}) \to (M, J_M)\) is \((J_{\Sigma}, J_M)\)-holomorphic. That is,

\[
J_M \circ F^* = F^* \circ J_{\Sigma}.
\]

For any compatible symplectic form \(\omega\) and associated Riemannian metric \(\bar{g}\) defined by (1.1), the induced metric and induced 2-form on \(\Sigma\) are given by

\[
g = F^* \bar{g}, \quad \omega = F^* \bar{\omega}.
\]

The cosine of the Kähler angle \(\alpha\) is define by (1.2):

\[
\omega = \cos \alpha d\mu_g.
\]

The following facts follow easily from (2.1): for any \(X, Y \in T\Sigma\),

(a) \(\omega(X, Y) = \bar{\omega}(F_*X, F_*Y) = \bar{g}(J_M F_*X, F_*Y) = \bar{g}(F_*J_{\Sigma}X, F_*Y) = g(J_{\Sigma}X, Y);\)

(b) \(g(J_{\Sigma}X, J_{\Sigma}Y) = \bar{g}(F_*J_{\Sigma}X, F_*J_{\Sigma}Y) = \bar{g}(J_M F_*X, J_M F_*Y) = \bar{g}(F_*X, F_*Y) = g(X, Y).\)

Now take any \(g\)-orthonormal basis \(\{e_1, e_2\}\) of \(T_p \Sigma\) for \(p \in \Sigma\). Then by definition and (a),

\[
\cos \alpha(p) = \omega(e_1, e_2) = g(J_{\Sigma}e_1, e_2).
\]

By (b), \(g(J_{\Sigma}e_1, e_1) = 0\). Thus, we have

\[
J_{\Sigma}e_1 = g(J_{\Sigma}e_1, e_2)e_2 = \cos \alpha e_2.
\]

Therefore,

\[
g(J_{\Sigma}e_1, J_{\Sigma}e_1) = \cos^2 \alpha(p) g(e_2, e_2) = \cos^2 \alpha(p).
\]

On the other hand, by (b),

\[
g(J_{\Sigma}e_1, J_{\Sigma}e_1) = g(e_1, e_1) = 1.
\]

Therefore, \(\cos^2 \alpha(p) = 1\), i.e., \(\sin^2 \alpha(p) = 0\). As \(p\) is arbitrary, we must have \(\sin \alpha \equiv 0\) on \(\Sigma\).
On the contrary, suppose \( \sin \alpha \equiv 0 \). As \( \cos \alpha \) is smooth on \( \Sigma \), without loss of generality, we can assume that \( \cos \alpha \equiv 1 \). Take any \( g \)-orthonormal basis \( \{e_1, e_2\} \) of \( T_p \Sigma \) for \( p \in \Sigma \). By (1.2), we have

\[ \omega(e_1, e_2) = d\mu_g(e_1, e_2) = 1. \]

On the other hand, by (1.1), we have

\[ \omega(e_1, e_2) = \bar{\omega}(F_s e_1, F_s e_2) = \bar{g}(J_M F_s e_1, F_s e_2). \]

Therefore, we have

(2.2) \[ \bar{g}(J_M F_s e_1, F_s e_2) = 1. \]

By the choice of \( \{e_1, e_2\} \), we get that

(2.3) \[ \bar{g}(F_s e_1, F_s e_j) = g(e_i, e_j) = \delta_{ij}. \]

Note that \( F_{sp}(T_p \Sigma) \) is a two-plane contained in \( T_{F(p)} M \). We denote \( N_{F(p)} \Sigma \) the normal space of \( F_{sp}(T_p \Sigma) \) in \( T_{F(p)} M \). Take a \( g \)-orthonormal basis of \( T_{F(p)} M \) \( \{F_s e_1, F_s e_2, e_3, \ldots, e_{2n}\} \), such that \( N_{F(p)} \Sigma = \text{span} \{e_3, \ldots, e_{2n}\} \). Then by (2.2),

\[ J_M F_s e_1 = \bar{g}(J_M F_s e_1, F_s e_1)F_s e_1 + \bar{g}(J_M F_s e_1, F_s e_2)F_s e_2 + \sum_{\alpha=3}^{2n} \bar{g}(J_M F_s e_1, e_\alpha)e_\alpha. \]

As by (2.3),

\[ \bar{g}(J_M F_s e_1, J_M F_s e_1) = 1 = \bar{g}(F_s e_2, F_s e_2), \]

we must have

(2.4) \[ J_M F_s e_1 = F_s e_2, \]

and thus

(2.5) \[ J_M F_s e_2 = -F_s e_1. \]

This implies that \( J_M \) maps \( F_{sp}(T_p \Sigma) \) onto itself. Therefore, \( F : \Sigma \rightarrow (M, J_M) \) is \( J_M \)-holomorphic.

\[ \text{Q.E.D.} \]

**Corollary 2.2.** Let \( (M^{2n}, \bar{\omega}, J_M, \bar{g}) \) be a compact Kähler manifold. If \( F : \Sigma \rightarrow (M, J_M) \) is a \( J_M \)-holomorphic immersion, then \( L^p \)-functional is constant on \( \mathcal{H} \) for any \( p \in \mathbb{Z} \). In particular any \( \rho \in \mathcal{H} \) is both a critical point and a stable point for the functional \( L^p \).

**Proof:** By Proposition 2.1, for each \( \rho \in \mathcal{H} \), \( \cos^2 \alpha_\rho \equiv 1 \) on \( \Sigma \). As \( \cos \alpha_\rho \) is smooth on \( \Sigma \), without loss of generality, we may assume that \( \cos \alpha_\rho \equiv 1 \). Then,

\[ L^p(\rho) = \int_{\Sigma} \cos^p \alpha_\rho d\mu_\rho = \int_{\Sigma} \cos \alpha_\rho d\mu_\rho = \int_{\Sigma} F^* \bar{\omega}_\rho = [\bar{\omega}_\rho][\Sigma] = [\bar{\omega}][\Sigma], \]

which is independent of \( \rho \in \mathcal{H} \).

\[ \text{Q.E.D.} \]
3. Critical Point and $J_M$-Holomorphicity

In this and next sections, we will consider the converse of Corollary 2.2. In this section, we compute the first variations of the Kähler angle and $L_p$-functional when we move the Kähler form on $M$ in the fixed Kähler class. We will prove that if $L_p$-functional has a critical point, then the immersion is $J_M$-holomorphic.

Let $(M^{2n}, \tilde{\omega}, J_M, \tilde{g})$ be a compact Kähler manifold. Given $\varphi \in \mathcal{H}$, define

\begin{equation}
\tilde{\omega}_\varphi = \tilde{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi.
\end{equation}

The associated Kähler metric is given by

\begin{equation}
\tilde{g}_\varphi(U, V) = \tilde{\omega}_\varphi(U, J_M V).
\end{equation}

Given the immersion $F : \Sigma \to M$, we have the induced metric and 2-form on $\Sigma$:

\begin{equation}
g_\varphi = F^* \tilde{g}_\varphi, \quad \omega_\varphi = F^* \tilde{\omega}_\varphi.
\end{equation}

The cosine of the Kähler angle $\alpha_\varphi$ is defined by

\begin{equation}
\omega_\varphi = \cos \alpha_\varphi d\mu_{g_\varphi}.
\end{equation}

Fix a $g$-orthonormal basis $\{e_1, e_2\}$ of $T_p \Sigma$, where $g = g_0$. Then by (1.2),

\begin{equation}
\cos \alpha = \omega(e_1, e_2).
\end{equation}

By (3.4), we have

\begin{equation}
\cos \alpha_\varphi = \frac{\omega_\varphi(e_1, e_2)}{\sqrt{\det(g_\varphi(e_i, e_j))}}.
\end{equation}

By (3.1) and (3.3), we have

\begin{equation}
\omega_\varphi = F^*(\tilde{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi) = \omega + F^*(\sqrt{-1} \partial \bar{\partial} \varphi),
\end{equation}

so that

\begin{equation}
\omega_\varphi(e_1, e_2) = \cos \alpha + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_1, F_* e_2).
\end{equation}

On the other hand, by the choice of frame and (3.1), (3.2), (3.3), we have

\begin{align*}
g_\varphi(e_i, e_j) &= \tilde{g}_\varphi(F_* e_i, F_* e_j) = \tilde{\omega}_\varphi(F_* e_i, J_M F_* e_j) \\
&= \tilde{\omega}(F_* e_i, J_M F_* e_j) + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_i, J_M F_* e_j) \\
&= \delta_{ij} + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_i, J_M F_* e_j).
\end{align*}

Namely,

\begin{align*}
(g_\varphi) &= \begin{pmatrix}
1 + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_1, J_M F_* e_1) & (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_1, J_M F_* e_2) \\
(\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_2, J_M F_* e_1) & 1 + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_2, J_M F_* e_2)
\end{pmatrix}.
\end{align*}

Therefore,

\begin{align*}
\det(g_\varphi) &= 1 + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_1, J_M F_* e_1) + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_2, J_M F_* e_2) \\
&\quad + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_1, J_M F_* e_1)(\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_2, J_M F_* e_2) \\
&\quad - [(\sqrt{-1} \partial \bar{\partial} \varphi)(F_* e_1, J_M F_* e_2)]^2.
\end{align*}
From (3.6) and (3.8), we have
\[(3.11) \quad \cos \alpha_{\varphi} = \frac{\cos \alpha + (\sqrt{-1} \partial \bar{\partial} \varphi)(F_{*}e_1, F_{*}e_2)}{\sqrt{\det(g_{\varphi}(e_i, e_j))}}.\]

It is known that (Lemma 2.1 of [1]), for any \(U, V \in TM\), we have
\[(3.12) \quad (\sqrt{-1} \partial \bar{\partial} \varphi)(U, V) = \frac{1}{2} \left[ -(\nabla^2 \varphi)(U, JMV) + (\nabla^2 \varphi)(V, JMU) \right], \]
and thus
\[(3.13) \quad (\sqrt{-1} \partial \bar{\partial} \varphi)(U, JMV) = \frac{1}{2} \left[ (\nabla^2 \varphi)(U, V) + (\nabla^2 \varphi)(JMU, JMV) \right]. \]

For simplicity, from now on, we will identify \(e_i\) with \(F_{*}e_i\). At a fixed point \(p \in \Sigma\), it is easy to see that we can choose a \(\bar{g}\)-orthonormal frame \(\{e_1, e_2, \cdots, e_{2n}\}\) of \(T_p\Sigma\), such that \(\{e_1, e_2\}\) spans the tangent space of \(\Sigma\), \(\{e_3, \cdots, e_{2n}\}\) spans the normal space of \(\Sigma\), and the complex structure takes the form
\[(3.14) \quad J_{M} = \begin{pmatrix} (J_1)_{4 \times 4} & 0_{4 \times (2n-4)} \\ 0_{(2n-4) \times 4} & (J_2)_{(2n-4) \times (2n-4)} \end{pmatrix}, \]
where
\[(3.15) \quad J_1 = \begin{pmatrix} 0 & \cos \alpha & \sin \alpha & 0 \\ -\cos \alpha & 0 & 0 & -\sin \alpha \\ -\sin \alpha & 0 & 0 & \cos \alpha \\ 0 & \sin \alpha & -\cos \alpha & 0 \end{pmatrix}, \]
and \(J_2\) satisfies \(J_2^2 = -Id_{2n-4}\).

By (3.12) and (3.14), we have
\[(3.16) \quad (\sqrt{-1} \partial \bar{\partial} \varphi)(e_1, e_2) = \frac{1}{2} \left\{ -(\nabla^2 \varphi)(e_1, JMe_2) + (\nabla^2 \varphi)(e_2, JMe_1) \right\} \]
\[= \frac{1}{2} \left\{ \cos \alpha [(\nabla^2 \varphi)(e_1, e_1) + (\nabla^2 \varphi)(e_2, e_2)] \right\} + \sin \alpha [(\nabla^2 \varphi)(e_1, e_4) + (\nabla^2 \varphi)(e_2, e_3)] \right\} \right\}.
\]

By (3.13) and (3.14), we have
\[(3.17) \quad (\sqrt{-1} \partial \bar{\partial} \varphi)(e_1, JMe_1) = \frac{1}{2} \left\{ (\nabla^2 \varphi)(e_1, e_1) + (\nabla^2 \varphi)(JMe_1, JMe_1) \right\} \]
\[= \frac{1}{2} \left\{ (\nabla^2 \varphi)(e_1, e_1) + \cos^2 \alpha (\nabla^2 \varphi)(e_2, e_2) \right\} + 2 \sin \alpha \cos \alpha (\nabla^2 \varphi)(e_2, e_3) + \sin^2 \alpha (\nabla^2 \varphi)(e_3, e_3) \right\},
\]
\[(3.18) \quad (\sqrt{-1} \partial \bar{\partial} \varphi)(e_2, JMe_2) = \frac{1}{2} \left\{ (\nabla^2 \varphi)(e_2, e_2) + (\nabla^2 \varphi)(JMe_2, JMe_2) \right\} \]
\[= \frac{1}{2} \left\{ (\nabla^2 \varphi)(e_2, e_2) + \cos^2 \alpha (\nabla^2 \varphi)(e_1, e_1) \right\} + 2 \sin \alpha \cos \alpha (\nabla^2 \varphi)(e_1, e_4) + \sin^2 \alpha (\nabla^2 \varphi)(e_4, e_4) \right\}.\]
and
\[
(\sqrt{-1}\partial\bar{\partial}\varphi)(e_1, J_M e_2) = \frac{1}{2} \left\{ (\nabla^2 \varphi)(e_1, e_2) + (\nabla^2 \varphi)(J_M e_1, J_M e_2) \right\} \\
= \frac{1}{2} \left\{ \sin^2 \alpha [(\nabla^2 \varphi)(e_1, e_2) - (\nabla^2 \varphi)(e_3, e_4)] \\
- \sin \alpha \cos \alpha [(\nabla^2 \varphi)(e_1, e_3) + (\nabla^2 \varphi)(e_2, e_4)] \right\}.
\]
(3.19)

We will compute the first variation of the Kähler angle. Let \( \varphi(t) \) be a family of Kähler potential on \( M \) so that \( \varphi(0) \equiv 0 \) and \( \dot{\varphi} = \psi \). Then by (3.10), we have
\[
d^{\partial t} \big|_{t=0} \det(g_{\varphi(t)}) = (\sqrt{-1}\partial\bar{\partial}\psi)(e_1, J_M e_1) + (\sqrt{-1}\partial\bar{\partial}\psi)(e_2, J_M e_2).
\]
(3.20)

By (3.11) and using (3.16), (3.17), (3.18) and (3.20), we have
\[
\frac{d}{dt} \big|_{t=0} \cos \alpha \varphi(t) \\
= (\sqrt{-1}\partial\bar{\partial}\psi)(e_1, e_2) - \frac{1}{2} \cos \alpha \frac{d}{dt} \big|_{t=0} \det(g_{\varphi(t)}) \\
= (\sqrt{-1}\partial\bar{\partial}\psi)(e_1, e_2) - \frac{1}{2} \cos \alpha [(\sqrt{-1}\partial\bar{\partial}\psi)(e_1, J_M e_1) + (\sqrt{-1}\partial\bar{\partial}\psi)(e_2, J_M e_2)] \\
= \frac{1}{2} \left\{ \cos \alpha [(\nabla^2 \psi)(e_1, e_1) + (\nabla^2 \psi)(e_2, e_2)] + \sin \alpha [(\nabla^2 \psi)(e_1, e_4) + (\nabla^2 \psi)(e_2, e_3)] \right\} \\
- \frac{1}{4} \cos \alpha \left\{ (1 + \cos^2 \alpha)(\nabla^2 \psi)(e_1, e_1) + (1 + \cos^2 \alpha)(\nabla^2 \psi)(e_2, e_2) \\
+ 2 \sin \alpha \cos \alpha (\nabla^2 \psi)(e_2, e_3) + \sin^2 \alpha (\nabla^2 \psi)(e_3, e_3) \\
+ 2 \sin \alpha \cos \alpha (\nabla^2 \psi)(e_1, e_4) + \sin^2 \alpha (\nabla^2 \psi)(e_4, e_4) \right\} \\
= \frac{1}{4} \sin^2 \alpha \left\{ \cos \alpha \left[ (\nabla^2 \psi)(e_1, e_1) + (\nabla^2 \psi)(e_2, e_2) - (\nabla^2 \psi)(e_3, e_3) - (\nabla^2 \psi)(e_4, e_4) \right] \\
+ 2 \sin \alpha \left[ (\nabla^2 \psi)(e_1, e_4) + (\nabla^2 \psi)(e_2, e_3) \right] \right\}.
\]
(3.21)

Now we can prove the first theorem in this section:

**Theorem 3.1.** Let \((M^{2n}, \bar{\omega}, J_M, \bar{g})\) be a Kähler manifold as above and \( F : \Sigma^2 \to M \) be a symplectic immersion. Set \( d : M \to \mathbb{R} \) any smooth extension from a tubular neighborhood of \( F(\Sigma) \) to \( M \) of the distance function from \( F(\Sigma) \), i.e. \( d(Q) = \text{dist}(Q, F(\Sigma)) \) for \( Q \) sufficiently near \( F(\Sigma) \). If
\[
\frac{d}{dt} \big|_{t=0} L_p(\bar{\omega}_p + t\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{2})) = 0
\]
for some \( p \in \mathbb{Z} - \{1\} \) and \( p \in \mathcal{H} \), then the immersion is \( J_M \)-holomorphic.

**Proof:** To simplify the notation, by assumption, without loss of generality, we assume that \( p \equiv 0 \) so that \( \bar{\omega}_p = \bar{\omega} \). Let \( \varphi(t) \) be any curve in \( \mathcal{H} \) so that \( \varphi(0) \equiv 0 \) and \( \dot{\varphi} = \psi \). At a fixed point \( p \in \Sigma \), take an orthonormal basis \( \{e_1, e_2\} \) of \( T_p \Sigma \) so that the complex structure
$J_M$ takes the form (3.14). By (3.9), it is easy to see that

\begin{equation}
(3.22) \quad \frac{d}{dt}|_{t=0}d\mu_t = \frac{1}{2} \sum_{i=1}^{2} (\sqrt{-1} \partial \bar{\partial} \psi)(e_i, J_M e_i) d\mu_t.
\end{equation}

By direct computation using (3.17) and (3.18), we obtain that

\begin{equation}
(3.23) \quad \frac{d}{dt}|_{t=0}d\mu_t = \frac{1}{4} \left\{ (1 + \cos^2 \alpha) (\nabla^2 \psi)(e_1, e_1) + (1 + \cos^2 \alpha) (\nabla^2 \psi)(e_2, e_2) + 2 \sin \alpha \cos \alpha (\nabla^2 \psi)(e_2, e_3) + \sin^2 \alpha (\nabla^2 \psi)(e_2, e_4) \right\}.
\end{equation}

By the definition of the functional $L_p$, (3.21) and (3.23), we compute

\begin{equation}
(3.24) \quad \frac{d}{dt}|_{t=0}L_p(\varphi(t)) = \int_{\Sigma} \cos^{p-1} \alpha \frac{d}{dt}|_{t=0} \cos \alpha \varphi(t) d\mu + \int_{\Sigma} \cos^p \alpha \frac{d}{dt}|_{t=0} d\mu_t
\end{equation}

Next, we will take special test function $\psi$. Actually, we will take the same $\psi$ as in the proof of Theorem 2.4 of [1]. By Corollary 2.2, to prove the theorem, it suffices to show that $\sin \alpha \equiv 0$ on $\Sigma$. We identify $\Sigma$ with its image in $M$. Denote $d$ the distance function of $M$ from $\Sigma$ with respect to the metric $g$. Namely, for $Q \in M$, $d(Q) = dist_g(Q, \Sigma)$. Then it is known that $\xi = \frac{1}{2}d^2$ is smooth in a neighborhood of $\Sigma$ in $M$, and for any $x_0 \in \Sigma$, $Hess(\xi)(x_0)$ represents the orthogonal projection on the normal space to $\Sigma$ at $x_0$. (See Proposition 2.5 of [1],) Namely, for each $U, V \in T_{x_0}M$ and $x_0 \in \Sigma$, we have

\begin{equation}
(3.25) \quad (\nabla^2 \xi)(U, V)(x_0) = \langle U^\perp, V^\perp \rangle,
\end{equation}

where $T_{x_0}M = T_{x_0}\Sigma \oplus N_{x_0}\Sigma$ and $U^\perp$ is the projection of $U$ onto $N_{x_0}\Sigma$. We take $\psi$ to be a smooth function on $M$ such that $\psi = \xi = \frac{1}{2}d^2$ in a neighborhood of $\Sigma$ in $M$. Then by
(3.25), for \( \tilde{\omega}(t) = \tilde{\omega} + t\sqrt{-1}\partial\bar{\partial}\psi \), we have from (3.24) that
\[
\frac{d}{dt}|_{t=0}L(\tilde{\omega} + t\sqrt{-1}\partial\bar{\partial}\psi) = \frac{1-p}{2} \int_{\Sigma} \sin^{2}\alpha \cos^{p} \alpha d\mu.
\]
By our assumption, \( \cos \alpha > 0 \), \( p \neq 1 \) and \( \frac{d}{dt}|_{t=0}L(\tilde{\omega} + t\sqrt{-1}\partial\bar{\partial}\psi) = 0 \). Therefore, we must have \( \sin \alpha \equiv 0 \). This proves the theorem.

Corollary 3.2. Let \((M^{2n}, \tilde{\omega}, J_{M}, \bar{g})\) be a Kähler manifold as above and \( F : \Sigma^{2} \rightarrow M \) be a symplectic immersion. If for some \( p \in \mathbb{Z} - \{1\} \), the functional \( L_{p} \) has a critical point in \( \mathcal{H} \), then the immersion is \( J_{M} \)-holomorphic.

When \( p = 0 \), the integrand of the right hand side of (3.26) becomes \( \frac{1}{2} \sin^{2} \alpha \). In this case, we do not need the immersion to be “symplectic”, and the theorem reduces to Theorem 1.2 of [1].

Theorem 3.3. ([1]) Let \((M^{2n}, \tilde{\omega}, J_{M}, \bar{g})\) be a Kähler manifold as above and \( F : \Sigma^{2} \rightarrow M \) be an immersion. Set \( d : M \rightarrow \mathbb{R} \) any smooth extension from a tubular neighborhood of \( F(\Sigma) \) to \( M \) of the distance function from \( F(\Sigma) \), i.e. \( d(Q) = \text{dist}(Q, F(\Sigma)) \) for \( Q \) sufficiently near \( F(\Sigma) \). If
\[
\frac{d}{dt}|_{t=0} L_{0}(\tilde{\omega}_{\rho} + t\sqrt{-1}\partial\bar{\partial}(\frac{d^{2}}{2})) = 0,
\]
then the immersion is \( J_{M} \)-holomorphic.

Corollary 3.4. ([1]) Let \((M^{2n}, \tilde{\omega}, J_{M}, \bar{g})\) be a Kähler manifold as above and \( F : \Sigma^{2} \rightarrow M \) be an immersion. If the functional \( L_{0} \) has a critical point in \( \mathcal{H}_{0} \), then the immersion is \( J_{M} \)-holomorphic.

When \( p > 0 \), we can say more. In this case, we allow \( \cos \alpha \) to have zeros in the definition of the functional \( L_{p} \). Obviously, a Lagrangian immersion (i.e., \( \cos \alpha \equiv 0 \)) achieves the minimum of the functional \( L_{p} \), while a holomorphic immersion achieves the maximum when \( p \) is even. Therefore, in this case, both Lagrangian immersion and holomorphic immersion have critical points. We will show that the converse is also true.

Theorem 3.5. Let \((M^{2n}, \tilde{\omega}, J_{M}, \bar{g})\) be a Kähler manifold as above and \( F : \Sigma^{2} \rightarrow M \) be an immersion. Set \( d : M \rightarrow \mathbb{R} \) any smooth extension from a tubular neighborhood of \( F(\Sigma) \) to \( M \) of the distance function from \( F(\Sigma) \), i.e. \( d(Q) = \text{dist}(Q, F(\Sigma)) \) for \( Q \) sufficiently near \( F(\Sigma) \). If
\[
\frac{d}{dt}|_{t=0} L_{p}(\tilde{\omega}_{\rho} + t\sqrt{-1}\partial\bar{\partial}(\frac{d^{2}}{2})) = 0
\]
for some \( p \in 2\mathbb{Z}^{+} \) and \( \rho \in \tilde{\mathcal{H}} \), then the immersion is \( J_{M} \)-holomorphic or Lagrangian.

**Proof:** Proceeding in the same way as in the proof of Theorem 3.1, we finally obtain (3.26). As \( p \) is a positive even integer, by the assumption, we must have
\[
\sin^{2} \alpha \cos^{p} \alpha \equiv 0, \text{ on } \Sigma.
\]
Denote
\[
\Omega_{1} = \{ x \in \Sigma : \sin \alpha(x) = 0 \}, \quad \Omega_{2} = \{ x \in \Sigma : \cos \alpha(x) = 0 \}.
\]
Then we have that: \( \Sigma = \Omega_{1} \cup \Omega_{2} \), \( \Omega_{1} \cap \Omega_{2} = \emptyset \) and both \( \Omega_{1} \) and \( \Omega_{2} \) are closed set. As \( \Sigma \) is connected, we must have \( \Omega_{1} = \emptyset \) or \( \Omega_{1} = \Sigma \).

If \( \Omega_{1} = \emptyset \), then \( \Omega_{2} = \Sigma \), i.e., \( \cos \alpha \equiv 0 \) on \( \Sigma \). In this case, the immersion is Lagrangian.
If $\Omega_1 = \Sigma$, i.e., $\sin \alpha \equiv 0$ on $\Sigma$, then by Proposition 2.1, the immersion is $J_M$-holomorphic. Q.E.D.

**Corollary 3.6.** Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If for some $p \in 2\mathbb{Z}^+$, the functional $L_p$ has a critical point in $\mathcal{H}$, then the immersion is $J_M$-holomorphic or Lagrangian.

### 4. Stable Point and $J_M$-Holomorphicity

In this section, we will compute the second variation formula for the functional $L_p$ and prove that existence of stable point implies $J_M$-holomorphicity. Instead of computing the variations of $\cos \alpha$ and $d\mu$ separately, we will compute the variations by combining them together.

Let $(M, \bar{\omega}, J_M, \bar{g})$ be as in the previous section and $F : \Sigma \to M$ be an immersion. (As before, we assume $F$ to be symplectic if $p < 0$.) At a fixed point, we can choose the local coordinate $\{x_1, x_2\}$ on $\Sigma$ such that at that point, $\{\partial_{x_1}, \partial_{x_2}\}$ is $g$-orthonormal. Using (3.11), we can rewrite $L$ as

$$(4.1) \quad L_p(\varphi) = \int_\Sigma (\cos \alpha + (\sqrt{-1} \bar{\partial} \partial \varphi) (e_1, e_2))^p \det(g_{\varphi}) \frac{1-p}{2} dx_1 \wedge dx_2.$$ 

Now take any curve $\varphi(t)$ in $\mathcal{H}$ with $\varphi(0) \equiv 0$, $\varphi = \psi$ and $\dot{\varphi} = \eta$. Denote

$$(4.2) \quad \nu_p(t) = (\cos \alpha + (\sqrt{-1} \bar{\partial} \partial \varphi) (e_1, e_2))^p \det(g_{\varphi}) \frac{1-p}{2}.$$ 

Then, we have

$$\frac{d}{dt} \nu_p(t) = p (\cos \alpha + (\sqrt{-1} \bar{\partial} \partial \varphi) (e_1, e_2))^p \det(g_{\varphi}) \frac{1-p}{2} \frac{d}{dt} (\sqrt{-1} \bar{\partial} \partial \varphi(t)) (e_1, e_2)$$

$$+ \frac{1-p}{2} (\cos \alpha + (\sqrt{-1} \bar{\partial} \partial \varphi) (e_1, e_2))^p \det(g_{\varphi}) \frac{1-p}{2} \frac{d}{dt} \det(g_{\varphi(t)}).$$

Hence, by using (3.10) and (3.20), we have

$$\frac{d^2}{dt^2} |_{t=0} \nu_p(t)$$

$$= p \cos^{p-1} \alpha \frac{d^2}{dt^2} |_{t=0} (\sqrt{-1} \bar{\partial} \partial \varphi(t)) (e_1, e_2)$$

$$+ (1-p) \cos^{p-1} \alpha \frac{d}{dt} |_{t=0} (\sqrt{-1} \bar{\partial} \partial \varphi(t)) (e_1, e_2) \frac{d}{dt} |_{t=0} \det(g_{\varphi(t)})$$

$$+ p(1-p) \cos^{p-2} \alpha \left[ \left. \frac{d}{dt} \right|_{t=0} (\sqrt{-1} \bar{\partial} \partial \varphi(t)) (e_1, e_2) \right]^2$$

$$+ \frac{(p-1)(p+1)}{4} \cos^p \alpha \left[ \left. \frac{d}{dt} \right|_{t=0} \det(g_{\varphi(t)}) \right]^2 + \frac{1-p}{2} \cos^p \alpha \frac{d^2}{dt^2} |_{t=0} \det(g_{\varphi(t)})$$

$$= p \cos^{p-1} \alpha (\sqrt{-1} \bar{\partial} \partial \eta) (e_1, e_2)$$

$$+ (1-p) \cos^{p-1} \alpha (\sqrt{-1} \bar{\partial} \partial \psi)(e_1, e_2) [(\sqrt{-1} \bar{\partial} \partial \psi)(e_1, J_M e_1) + (\sqrt{-1} \bar{\partial} \partial \psi)(e_2, J_M e_2)]$$

$$+ p(1-p) \cos^{p-2} \alpha [(\sqrt{-1} \bar{\partial} \partial \psi)(e_1, e_2)]^2.$$
$$+ \frac{(p-1)(p+1)}{4} \cos^p \alpha \left[ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2) \right]^2$$

$$+ \frac{1-p}{2} \cos^p \alpha \left\{ (\sqrt{-1} \partial \bar{\partial} \eta)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \eta)(e_2, J_M e_2) \right\}$$

$$+ 2(\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1)(\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2) - 2 \left[ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_2) \right]^2$$

$$= \cos^p \alpha \left\{ \frac{1-p}{2} \left[ (\sqrt{-1} \partial \bar{\partial} \eta)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \eta)(e_2, J_M e_2) \right] + \frac{p(\sqrt{-1} \partial \bar{\partial} \eta)(e_1, e_2)}{\cos \alpha} \right\}$$

$$+ p(1-p) \cos^{p-1} \alpha (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, e_2)[(\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2)]$$

$$+ p(p-1) \cos^{p-2} \alpha \left[ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, e_2) \right]^2 \mu + p \int_{\Sigma} \cos^p \alpha \left[ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2) \right]^2 \mu$$

$$+ \frac{(p-1)(p+1)}{4} \int_{\Sigma} \cos^p \alpha \left\{ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2) \right\}^2 \mu + (1-p) \int_{\Sigma} \cos^p \alpha \left\{ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1)(\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2) - [ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_2) ]^2 \right\} \mu.$$ (4.3)

Therefore, the second variation formula for the functional $L_p$ is given by

$$\frac{d^2}{dt^2} |_{t=0} L_p(\varphi(t))$$

$$= \int_{\Sigma} \cos^p \alpha \left\{ \frac{1-p}{2} \left[ (\sqrt{-1} \partial \bar{\partial} \eta)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \eta)(e_2, J_M e_2) \right] + \frac{p(\sqrt{-1} \partial \bar{\partial} \eta)(e_1, e_2)}{\cos \alpha} \right\} \mu$$

$$+ p(1-p) \int_{\Sigma} \cos^{p-1} \alpha (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, e_2)[(\sqrt{-1} \partial \bar{\partial} \psi)(e_1, J_M e_1) + (\sqrt{-1} \partial \bar{\partial} \psi)(e_2, J_M e_2)] \mu$$

$$+ p(p-1) \int_{\Sigma} \cos^{p-2} \alpha \left[ (\sqrt{-1} \partial \bar{\partial} \psi)(e_1, e_2) \right]^2 \mu$$

Our main result in this section is as follows:

**Theorem 4.1.** Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be a symplectic immersion. Set $d : M \to \mathbb{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to $M$ of the distance function from $F(\Sigma)$, i.e. $d(Q) = \text{dist}(Q, F(\Sigma))$ for $Q$ sufficiently near $F(\Sigma)$. If

$$\frac{d^2}{dt^2} |_{t=0} L_p(\bar{\omega}_p + t^2 \sqrt{-1} \partial \bar{\partial} \left( \frac{d^2}{dt^2} \right)) = 0$$

for some $p \in \mathbb{Z} \setminus \{1\}$ and $\rho \in \mathcal{H}$, then the immersion is $J_M$-holomorphic.

**Proof:** To simplify the notation, by assumption, without loss of generality, we assume that $\rho \equiv 0$ so that $\bar{\omega}_p = \bar{\omega}$. As before, we will take special variations to prove that $\sin \alpha \equiv 0$. We take $\varphi(t) = t^2 \eta$ so that $\psi \equiv 0$. Then the second variation formula (4.3) becomes

$$\frac{d^2}{dt^2} |_{t=0} L_p(\varphi(t))$$
Now, we take \( \eta \) to be a smooth function on \( M \) so that \( \eta = \xi = \frac{d^2}{dt^2} \) in a neighborhood of \( \Sigma \). Here \( \xi \) is the function appearing in the proof of Theorem 3.1. By (3.25) and the expressions (3.16), (3.17) and (3.18), we see that when restricting on \( \Sigma \), we have

\[
(\sqrt{-1}\partial\bar{\partial}\eta)(e_1, e_2) = 0,
\]

and

\[
(\sqrt{-1}\partial\bar{\partial}\eta)(e_1, J_M e_1) = (\sqrt{-1}\partial\bar{\partial}\eta)(e_2, J_M e_2) = \frac{1}{2}\sin^2\alpha.
\]

Therefore, for this choice of \( \eta \), we have

\[
\frac{d^2}{dt^2}|_{t=0}L_{\rho}(\bar{\omega}_\rho + t^2\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{4})) = \frac{1}{2}\int_{\Sigma}\sin^2\alpha \cos\rho \alpha d\mu.
\]

By our assumption, \( \cos\alpha > 0 \), \( p \neq 1 \) and \( \frac{d^2}{dt^2}|_{t=0}L(\bar{\omega}_\rho + t^2\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{4})) = 0 \). Therefore, we must have \( \sin\alpha \equiv 0 \). This proves the theorem. \( \text{Q.E.D.} \)

**Corollary 4.2.** Let \((M^{2n}, \bar{\omega}, J_M, \bar{g})\) be a Kähler manifold as above and \( F: \Sigma^2 \to M \) be a symplectic immersion. If the functional \( L_p \) \((p \in \mathbb{Z} - \{1\})\) has a stable point in \( \mathcal{H} \), then the immersion is \( J_M \)-holomorphic.

**Proof:** As before without loss of generality, we can assume that \( L_p \) has a stable point \( \rho \equiv 0 \). Then by Definition 1.2, we see that for any \( \eta \in C^\infty(M, \mathbb{R}) \), we have

\[
\frac{d^2}{dt^2}|_{t=0}L_{\rho}(\bar{\omega} + \frac{t^2}{2}\sqrt{-1}\partial\bar{\partial}\eta) \geq 0.
\]

By (4.4), we can see that

\[
\frac{d^2}{dt^2}|_{t=0}L_{\rho}(\bar{\omega} - \frac{t^2}{2}\sqrt{-1}\partial\bar{\partial}\eta) = -\frac{d^2}{dt^2}|_{t=0}L_{\rho}(\bar{\omega} + \frac{t^2}{2}\sqrt{-1}\partial\bar{\partial}\eta).
\]

Replacing \( \eta \) by \(-\eta\) in (4.5), we can obtain that

\[
\frac{d^2}{dt^2}|_{t=0}L_{\rho}(\bar{\omega} + \frac{t^2}{2}\sqrt{-1}\partial\bar{\partial}\eta) = 0,
\]

for any \( \eta \in C^\infty(M, \mathbb{R}) \). Now the Corollary follows immediately form Theorem 4.1. \( \text{Q.E.D.} \)

As in Section 3, we can remove the "symplectic" assumption for \( p = 0 \) ([1]):

**Theorem 4.3.** Let \((M^{2n}, \bar{\omega}, J_M, \bar{g})\) be a Kähler manifold as above and \( F: \Sigma^2 \to M \) be an immersion. Set \( d: M \to \mathbb{R} \) any smooth extension from a tubular neighborhood of \( F(\Sigma) \) to \( M \) of the distance function from \( F(\Sigma) \), i.e. \( d(Q) = \text{dist}(Q, F(\Sigma)) \) for \( Q \) sufficiently near \( F(\Sigma) \). If

\[
\frac{d^2}{dt^2}|_{t=0}L_0(\bar{\omega}_\rho + t^2\sqrt{-1}\partial\bar{\partial}(\frac{d^2}{4})) = 0
\]

for some \( \rho \in \mathcal{H} \), then the immersion is \( J_M \)-holomorphic.

**Corollary 4.4.** Let \((M^{2n}, \bar{\omega}, J_M, \bar{g})\) be a Kähler manifold as above and \( F: \Sigma^2 \to M \) be an immersion. If the functional \( L_0 \) has a stable point in \( \mathcal{H} \), then the immersion is \( J_M \)-holomorphic.
When $p > 0$, arguing in the same way as in the proof of Theorem 3.5, we can obtain:

**Theorem 4.5.** Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. Set $d : M \to \mathbb{R}$ any smooth extension from a tubular neighborhood of $F(\Sigma)$ to $M$ of the distance function from $F(\Sigma)$, i.e. $d(Q) = \text{dist}(Q, F(\Sigma))$ for $Q$ sufficiently near $F(\Sigma)$. If

$$ \frac{d^2}{dt^2}|_{t=0} L_p(\bar{\omega}_\rho + t^2 \sqrt{-1} \partial \bar{\partial}(\frac{d^2}{4})) = 0 $$

for some $p \in 2\mathbb{Z}^+$ and $\rho \in \mathcal{H}$, then the immersion is $J_M$-holomorphic or Lagrangian.

**Corollary 4.6.** Let $(M^{2n}, \bar{\omega}, J_M, \bar{g})$ be a Kähler manifold as above and $F : \Sigma^2 \to M$ be an immersion. If for some $p \in 2\mathbb{Z}^+$, the functional $L_p$ has a stable point in $\mathcal{H}$, then the immersion is $J_M$-holomorphic or Lagrangian.

**References**


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